Chapter 1

Introduction

1.1 Porous Medium

A porous medium is a solid matrix containing *pores* either connected or non-connected dispersed within the medium in a regular or random manner. Intuitively *pores* are void spaces which must be distributed more or less frequently through the material if it is to be called *porous*. Extremely small voids in a solid are called *molecular interstices* while very large ones are called *caverns*. Pores are void spaces intermediate between caverns and molecular interstices. The pores in the porous medium may be *interconnected* or *non-interconnected*. Flow of intersticial fluid is possible only if at least part of the pore space is interconnected. The interconnected part of the pore system is called the *effective pore space* of the porous medium and the whole of the pore space is called the *total pore space*. In addition porous material is classified as *ordered porous material* and *random porous material* according as the pore spaces inside the porous material is ordered and disordered. According to the above description a few examples of porous media are: beds formed of sand, granules, lead shot, etc.; porous rocks such as lime stone, pumice, dolomite, etc.; fibrous aggregates such as cloth, filter paper, etc.

Porosity of a porous material is the fraction of void space, distributed through the material, to the total volume and is expressed either as a fraction of one or in per cent. This gives the total porosity of the medium. If the calculation of the porosity is based upon the interconnected pore space instead of the total pore space, the resulting value is termed as *effective porosity.* The permeability k is the most important physical property of a porous medium like the porosity is its most important geometrical property. Permeability measures quantitatively the ability of the porous medium to conduct fluid flow. Therefore the homogeneous-fluid-carrying capacity of a porous medium is completely described by its permeability and is only indirectly effected by the porosity, which is essentially a static property. The permeability depends on the microstructure of the medium and is independent of the properties of the saturated fluid.

1.2 Equation of Continuity

The most important equation of fluid flow is continuity equation which is nothing but simply a mathematical formulation of the principle of conservation of mass. For a control volume that has a single inlet and a single outlet, the principle of conservation of mass states that the mass flow rate into the volume must equal the mass flow rate out. This implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0, \tag{1.2.1}$$

where ρ is the density of the fluid and $\mathbf{V}$ is the area averaged velocity vector. For a steady homogeneous and incompressible fluid, ρ is constant throughout the entire fluid and hence the equation of continuity reduces to

$$\nabla \cdot \mathbf{V} = 0. \tag{1.2.2}$$

1.3 Equation of Motion

Viscous Flow

The central equations for fluid dynamics are the Navier-Stokes equations, which are non-linear partial differential equation that describes fluid flow given by

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho \mathrm{g}, \tag{1.3.1}$$

where $\mathbf{V}$ is the velocity of the fluid, μ is the viscosity of the fluid, ρ is the density and $\mathbf{g}$ is the body force. The class of incompressible viscous flows is assumed to have negligible inertial effects when the ratio of the inertial forces to the viscous forces, which is called Reynolds number ($Re = \frac{F_{inertial}}{F_{viscous}} = \frac{VL\rho}{\mu}$, L is the characteristic length) is sufficiently low. Therefore, in Navier-Stokes equations, viscous term will dominate the inertial term and hence neglecting the inertial term we get linear equations called Stokes equations given by

$$\mu\nabla^2\mathbf{V} = \nabla p, \tag{1.3.2}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{1.3.3}$$

where μ is the viscosity of the flow, $\mathbf{V}$ is the velocity and p is the pressure of the fluid. Therefore, the flow of homogeneous fluids at low Reynolds number governed by Stokes equations together with equation of continuity is considered as a Stokes flow.

The governing equation of laminar flow through homogeneous porous media is based on a classical experiment originally performed by Darcy [1]. He was interested in the flow characteristics of sand filters and so he had to do an experimental study of the problem and established the quantitative theory of the flow of homogeneous fluids through porous media. These experiments gave the very simple result, generally referred to as Darcy's law - that the rate of flow Q of fluid through the porous bed was directly proportional to the area A of the sand bed and to the difference Δh between the fluid heads at the inlet and outlet faces of the bed, and inversely proportional to the thickness L of the bed which can be expressed mathematically as

$$Q = \frac{k' A\Delta h}{L}. \tag{1.3.4}$$

He called the proportionality constant k' as the hydraulic conductivity. The pressure head h is equal to $(z+\frac{p}{\rho g})$ where z is the elevation, p is the pressure and ρ is the density of the fluid. Experiments show that the constant of proportionality of Darcy's law, hydraulic conductivity, is proportional to the density and inversely to the viscosity of the fluid. Expressing the above equation in terms of the space averaged velocity (or Darcian velocity), we have

$$v = -\frac{k}{\mu}\frac{d}{ds}(p+\rho g z), \tag{1.3.5}$$

where k is the permeability of the medium and is given by $k = \frac{\mu k'}{\rho g}$. The relation between the permeability k and the porosity ϕ of the medium is given by $k = \frac{d^2\phi^3}{(1-\phi)^2}$ for the porous medium composed of solid spheres, d being the diameter of the sphere. The negative sign in the above equation indicates that the fluid velocity is in the opposite direction of increasing pressure gradient. As this macroscopic flow is necessarily of one-dimensional character, it is necessary to generalize this result so that any type of flow system can be developed. Thus in a general three dimensional flow system, the resultant velocity at any point is directly proportional to the resultant pressure gradient in magnitude and in the same direction. Therefore the general form of the Darcy's law is

$$\mathbf{V} = -\frac{k}{\mu}(\nabla p - \mathbf{F}), \tag{1.3.6}$$

where $\mathbf{F}$ is the body force which can be replaced by the gravitational force $\mathbf{g} = (0, 0, -g)$. The permeability k is constant for an isotropic medium. This model does not take inertial effects into consideration and so this model is valid for seepage velocity only i.e., for flows with $O(Re) < 1$, where Re is the Reynolds number.

In Darcy's law, no viscous stress tensor has been defined. The viscous shearing stresses acting on a volume element of fluid have been neglected; only the damping force of the porous media ($\frac{\mu v}{k}$) has been retained which is a good approximation for small permeability. Brinkman found an equation which retains its validity for low particle density of porous media. As the fluid flow through porous media is described by Darcy's law, which is a first order differential equation and the flow outside the porous region is Stokes flow, which is a second order equation, it is impossible to formulate rational boundary conditions at the interface. Brinkman [2] suggested an equation stating the equilibrium between the forces acting on the volume element, i.e, pressure gradient, the divergence of the viscous stress tensor and the damping force caused by the porous mass. Hence, the Brinkman's equation for the fluid flow in porous media valid for large permeability is

$$\nabla p = -\frac{\mu}{k}\mathbf{V} + \mu' \nabla^2 \mathbf{V}, \tag{1.3.7}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{1.3.8}$$

where $\mathbf{V}$ is the rate of flow, μ is the fluid viscosity and μ' is the effective viscosity parameter. The so-called effective viscosity should not be thought of as the viscosity of the fluid but only a parameter, that allows for matching of the shear stress boundary condition across the free fluid-porous medium interface. But, in most of the cases it is assumed that $\mu' = \mu$. This equation has the advantage of approximating Darcy's law for small values of k and Stokes equation for large values of k.

There are several solutions of Stokes equation in the literature. To name a few, Lamb's solution [3], Lorentz solution [4], Besset's solution [5], Ranger's solution [6] etc. But each solution has its own disadvantage in the sense that some are not in closed form and some are not complete general solutions. A solution of the equations of motion, which is such that every other solution can be obtained from it is said to be a complete general solution. But, recently Palaniappan et al. [7] proposed a solution of Stokes equations which is given by

$$\mathbf{V} = CurlCurl(\mathbf{r}A) + Curl(\mathbf{r}B), \tag{1.3.9}$$

$$p = p_0 + \mu\frac{\partial}{\partial r}(r\nabla^2 A), \tag{1.3.10}$$

where p_0 is a constant, $\mathbf{r}$ is the position vector, A and B are scalars satisfying the equations

$$\nabla^4 A = 0, \tag{1.3.11}$$

$$\nabla^2 B = 0. \tag{1.3.12}$$

Later Padmavathi et al. [8] proved that the above representation is a complete general solution of Stokes equations.

While discussing the problem of Stokes flow past porous regions using Brinkman's equations, Higdon and Kojima [9] constructed the solutions for exterior and interior regions of porous particles. A cartesian tensor solution was also given by Yu and Kaloni [10]. Recently, Padmavathi et al. [11] gave a solution for the velocity and pressure in Brinkman's equations in terms of two scalar functions A and B, where

$$\mathbf{V} = CurlCurl(\mathbf{r}A) + Curl(\mathbf{r}B), \tag{1.3.13}$$

$$p = p_0 + \mu\frac{\partial}{\partial r}[r(\nabla^2 - \lambda^2)A], \tag{1.3.14}$$

where $\lambda^2 = \frac{1}{k}$, p_0 is a constant, A and B satisfy the equations

$$\nabla^2(\nabla^2 - \lambda^2)A = 0, \tag{1.3.15}$$

$$(\nabla^2 - \lambda^2)B = 0. \tag{1.3.16}$$