Chapter 1

Introduction

1.1 Introduction

The great mathematician Euler once remarked that "nothing at all takes place in the universe in which some rule of maximum or minimum does not appear". Though it is an extreme statement, it has been observed that a large class of problems in science, engineering, economic, etc. can be represented through the following optimization problem,

$$\begin{array}{lll} (P) & \min & f(x) \\ \text{subject to} & & g(x) \leq 0 \\ & & x \in M \end{array}$$

where f, $g = (g_1, \ldots, g_m)$ are real valued functions on R^n, $M \subseteq R^n$. The problem (P) as a whole is known as "Nonlinear Programming Problem" if at least one of f, $g_1, \ldots, g_m$ is a nonlinear function.

The central concept in optimization is known as the duality theory which asserts that, given a (primal) minimization problem, the infimum value of the primal minimization problem cannot be smaller than the supremum value of the associated (dual) maximization problem and the optimal values of the primal and dual problem are equal under certain conditions. Duality principle is based on the geometric relation, which states that the shortest distance from a point to a convex set is equal to the maximum of the distances from the point to a hyperplane separating the point from the set. Hence the original minimization problem over vectors can be converted to a maximization problem over hyperplanes. This result is more applicable in several computational applications where the choice is often made so that evaluating the dual maximum is comparatively easier than solving a primal minimization problem. In general, a dual problem for a nonlinear programming problem is not defined uniquely and duality is not a reciprocal relation. Thus except in the linear programming case, the dual of a dual need not be primal for general nonlinear programming problems. This has led to introduce several forms of dual for a nonlinear programming problem.

The convexity of sets and the convexity of functions have been the object of many investigations during the past century. This is mainly due to the development of the theory of mathematical programming, both linear and nonlinear, which is closely tied with convex analysis. Optimality conditions and duality were mainly established for classes of problems involving the optimization of convex objective functions over convex feasible regions. In the second half of the past century various generalizations of convex functions have been introduced. One of the most important generalization of convex functions are the invex functions which were initially employed by Hanson [48] and Craven [22]. Usually, generalized convex functions have been introduced in order to weaken as much as possible the convexity requirements for results related to optimization theory (in particular, optimality conditions and duality results). In the classical theory of optimization, the results on sufficient optimality conditions and duality are based on convexity assumption which is rather restrictive in application. An important feature in the use of convexity is that local optimal solution implies global optimal solution. But a number of problems arising in real world are not convex programming problem by nature. Generally it is not easy to handle such types of problems. Though it is analytically tough to solve these general nonconvex problems, yet many attempts are being made in this direction to justify zero duality gap between primal and dual nonlinear programming problems using generalized convexity assumptions by many researchers like Mangasarian, Mond, Bector, Hanson, Jeyakumar etc.

There are many real world situations where most of the information are imprecise in nature involving vagueness due to the presence of linguistic data. In the context of optimization problems the various parameters involved in the objective function or constraints or both have such type of uncertainties. These types of optimization problems can be handled by using fuzzy set theory introduced by Zadeh(1965). The use of fuzzy set theoretic concepts in optimization problem is known as fuzzy optimization. This is one of the emerging areas of application of fuzzy mathematics in decision science. The decision making problem in fuzzy environment was introduced by Bellman and Zadeh [9] in 1970. It has been developed by many researcher like Zimmermann [103,104], Tanake [87], Rammelfanger [78], Bector and Chandra [8] etc. in various directions. Very less work has been done to study the optimality condition and duality relationship for fuzzy linear and nonlinear programming problem.

This thesis deals with the optimality and duality results of general nonlinear programming problem and fuzzy nonlinear programming problem in following three broad categories.

1. Development of differentiable E-convexity, generalized E-convexity concept and its application in duality theory for general nonlinear programming problem. (Chapters 2, 3 and 4)

2. Study of duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual and second order duals results using convexity and E-convexity . (Chapter 5 and 6)

3. Study of duality results for fuzzy nonlinear programming problem using convexity and E-convexity . (Chapter 7 and 8)

Before we present the chapter wise brief summary of the results derived in the thesis, we collect some recent literature on optimization theory involving convex, invex, generalized convex, generalized invex functions, E-convex functions and some basic concepts of fuzzy set theory in next two sections.

1.2 Prerequisites

Karush [64] in 1939 and Kuhn and Tucker [68] in 1951 introduced Karush-Kuhn-Tucker(KKT) conditions which are necessary and sufficient condition for the existence of solution of a convex nonlinear programming problem. We state the KKT condition below.

Theorem 1.1. *(Necessary KKT Conditions) Consider the problem* (P) *with* M *open. Suppose* $f : R^n \to R$ *and* $g : R^n \to R^m$ *are real valued differentiable functions. Let* x_0 *be a local minimum to* (P)*. Moreover assume that* $\nabla g_i(x_0),\ i \in I$ *forms a linearly independent set of vectors in* R^n*. Then there exists* $u \in R^m_+$ *such that*

$$\begin{aligned} (i) \quad & \nabla f(x_0) + \sum_{i=1}^{m} u_i \nabla g_i(x_0) = 0 \\ (ii) \quad & u_i g_i(x_0) = 0. \end{aligned}$$

The KKT conditions are also sufficient if we assume that $f,\ g_1, \ldots, g_m$ are convex functions and M is a convex set.

Theorem 1.2. *(Sufficient KKT Conditions) Consider the problem* (P) *with* M *open. Suppose* $f : R^n \to R$ *and* $g : R^n \to R^m$ *are real valued differentiable functions.* f *and* g *are convex at* $x_0 \in M$*. If* $(x_0,\ u) \in M \times R^m_+$ *satisfy the following system*

$$\begin{aligned} (i) \quad & \nabla f(x_0) + \sum_{i=1}^{m} u_i \nabla g_i(x_0) = 0 \\ (ii) \quad & u_i g_i(x_0) = 0 \end{aligned}$$

then x *is local optimal solution of* (P)*.*

Definition 1.1 (80)**.** *A nonempty subset* M *of* R^n *is said to be convex set if* $(1-\lambda)x + \lambda y \in M$ *for all* $x, y \in M$ *and all* $\lambda \in [0, 1]$*.*

Definition 1.2 (80)**.** *Let* M *be a nonempty subset of* R^n*. A function* $f : M \to R$ *is said to be convex on* M *if* M *is a convex set and*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x,\ y \in M$ *and all* $\lambda \in [0, 1]$*.*

Definition 1.3 (80)**.** *A differentiable function is convex if and only if*

$$f(x) - f(y) \geq \nabla f(y)(x - y) \quad \forall \quad x, \; y \in M.$$

Definition 1.4 (70)**.** *Let $f : R^n \to R$ be a twice differentiable function on an open set $M \subset R^n$. f is convex on M if and only if $\nabla^2 f(x)$ is positive semidefinite on M, that is, for each $x \in M$*

$$y \nabla^2 f(x) y \geq 0 \quad \textit{for all } y \in R^n$$

Definition 1.5 (73)**.** *Let M be a nonempty open subset of R^n. Suppose $f : R^n \to R$ is a twice differentiable function M. Then f is said to be second order convex on M if*

$$f(x) - f(y) \geq (\nabla f(y))^T (x - y) + \frac{1}{2}(x - y)^T \nabla^2 f(y)(x - y) \quad \textit{for all } x, \; y \in M.$$

The first step in relaxing the convexity assumption was taken by Mangasarian [70], who introduced the notion of pseudoconvex and quasiconvex functions.

Definition 1.6 (70)**.** *A differentiable function $f : M \to R$ is called pseudoconvex if given $x, \; y \in M$*

$$\nabla f(y)(x - y) \geq 0 \Rightarrow f(x) \geq f(y)$$

and quasi-convex if

$$f(x) \leq f(y) \Rightarrow \nabla f(y)(x - y) \leq 0.$$

In 1981, Hanson [48] observed that term $(x - y)$, on the right hand side of the inequality in Definition 1.3, plays no role in the proof of the sufficiency of the Karush-Kuhn-Tucker conditions. This motivated Hanson [48] to introduce a class of differentiable functions $f : M \to R$ which satisfy

$$f(x) - f(y) \geq \nabla f(y) \eta(x, y) \quad \forall \quad x, \; y \in R^n,$$

where $\eta : R^n \times R^n \to R^n$ is any arbitrary function. If $\eta(x, y) = x - y$ then f becomes a convex function. Hanson [48] proved that the KKT conditions are also sufficient if the objective and constraint functions satisfy the above inequality with same function η. Later, Craven [20] called this class of functions as invex functions. Invex function was further revealed when Hanson [48] showed that invex functions could replace convex functions to prove the strong duality theorem for the Wolfe's dual. For defining an invex function we need the notion of differentiability. A relevant generalization of convex function without assuming differentiability is the following notion of a preinvex function which was first put forward by Ben-Israel and Mond [10] and later on by Weir and Mond [94], Hanson and Mond [51] and Pini [77].

Definition 1.7 (10)**.** *A set $M \subseteq R^n$ is said to be invex with respect to $\eta : R^n \times R^n \to R^n$ if $y + \lambda\eta(x, y) \in M, \quad \forall\ x,\ y \in M,\ \forall\ \lambda \in [0, 1]$.*

Definition 1.8 (10)**.** *Let $M \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. A function $f : M \to R$ is said to be preinvex with respect to η on M if*

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x,\ y \in M$ and all $\lambda \in [0, 1]$.

Another relevant generalization of convex sets and convex functions is the notion of E-convex set and E-convex functions which is defined by Youness [99] in 1999. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. Youness [99] established the optimality results for E-convex programming problems. Yang [98] and Chen [17] modified the results of Youness [99] on E-convex programming problem.

Definition 1.9 (99). *Let $E : R^n \to R^n$. A nonempty subset M of R^n is said to be E-convex set if $(1-\lambda)E(x) + \lambda E(y) \in M$ for all $x, y \in M$ and all $\lambda \in [0, 1]$.*

Definition 1.10 (99). *Let M be a nonempty subset of R^n and $E : R^n \to R^n$. A function $f : M \to R$ is said to be E-convex on M if M is an E-convex set and*

$$f((1-\lambda)E(x) + \lambda E(y)) \leq (1-\lambda)f(E(x)) + \lambda f(E(y))$$

for all $x,\ y \in M$ and all $\lambda \in [0, 1]$.

Lemma 1.1 (99). *If a set $M \subseteq R^n$ is E-convex, then $E(M) \subseteq M$.*

Chen [17] introduced a new class of semi-E-convex functions and studied some of its properties.

Definition 1.11 (17). *Let $M \subseteq R^n$ be an E-convex set and $E : R^n \to R^n$. A function $f : M \to R$ is said to be semi-E-convex on M if*

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x,\ y \in M$ and all $\lambda \in [0, 1]$.

Definition 1.12 (17). *Let $M \subseteq R^n$ be an E-convex set. A function $f : M \to R$ is said to be semi-E-quasiconvex on M if*

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \max\{f(x), f(y)\}$$

for all $x,\ y \in M$ and all $\lambda \in [0, 1]$.

Syau and Lee [86] defined E-quasi convex function, strictly E-quasi convex function and studied some basic properties.

Definition 1.13 (86). *Let $M \subseteq R^n$ be an E-convex set. A function $f : M \to R$ is said to be E-quasiconvex on M if*

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \max\{f(E(x)), f(E(y))\}$$

for all $x,\ y \in M$ and all $\lambda \in [0, 1]$, and strictly E-quasiconvex if strict inequality holds for all $x,\ y \in M$, $E(x) \neq E(y)$ and $\lambda \in [0, 1]$.

Lemma 1.2 (86). *Let f and g be E-convex functions on M and let $\alpha > 0$. Then $f + g$ and αf are E-convex functions on M.*

Lemma 1.3 (86). *Suppose that $E(M)$ is a convex set. Then $f : M \to R$ is E-convex (respectively strictly E-convex) if and only if its restriction, $\tilde{f} : E(M) \to R$ is a convex (respectively strictly convex) function.*

One of the most fruitful theories of duality in convex optimization is based on the concept of conjugate function. This concept was due to Fenchel [34] and Rockfellar [80] in the finite dimensional case. Here we present some basic results due to Rockafellar, Fenchel, Bazaraa, Sherali , Shetty et al. which are used in Chapters 5 and 6 to study the duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual.

Definition 1.14 (80). *The function $f^* : R^n \to R$, defined by*

$$f^*(p^*) = \sup_{x \in R^n} \{ {p^*}^T x - f(x) \}$$

is called as the conjugate function of $f : R^n \to R$.

Lemma 1.4 (80). *Let C_1 and C_2 be non-empty sets in R^n. There exists a hyperplane separating C_1 and C_2 properly if and only if there exists a vector b such that*
(a) $\inf\{x^T b | \ x \in C_1\} \geq \sup\{x^T b | \ x \in C_2\}$,
(b) $\sup\{x^T b | \ x \in C_1\} > \inf\{x^T b | \ x \in C_2\}$.
There exists a hyperplane separating C_1 and C_2 strongly if and only if there exists a vector b such that
(c) $\inf\{x^T b | \ x \in C_1\} > \sup\{x^T b | \ x \in C_2\}$.

Lemma 1.5 (80). *Let C_1 and C_2 be non-empty convex sets in R^n. In order that there exists a hyperplane separating C_1 and C_2 properly, it is necessary and sufficient that $rintC_1$ and $rintC_2$ have no point in common. ($rintC_1$ and $rintC_2$ are the relative interiors of C_1 and C_2)*

Lemma 1.6 (7). *Let S be a nonempty convex set in R^n and $\overline{x} \in intS$. Then there is a nonzero vector p such that $p^T(x - \overline{x}) \leq 0$ for each $x \in clS$.*

Lemma 1.7 (7). *Let X be a nonempty convex set in R^n. Let $\alpha : R^n \to R$ and $g : R^n \to R^m$ be convex, and let $h : R^n \to R$ be affine; that is, $h = Ax - b$. If System 1 below has no solution x, then System 2 has a solution (u_0, u, v). The converse holds if $u_0 > 0$.*
System 1: $\alpha(x) < 0 \quad g(x) \leq 0 \qquad h(x) = 0$ *for some* $x \in X$
System 2: $u_0\alpha(x) + u^T g(x) + v^T h(x) \geq 0$ *for all* $x \in X$
$(u_0, u) \geq 0, \quad (u_0, u, v) \neq 0.$

In Chapters 7 and 8, we have derived duality results for the fuzzy nonlinear programming problem (FNP). In a fuzzy nonlinear programming problem (FNP) fuzziness is present in the objective function or in the constraints or in both, either in terms of fuzzy variables or in terms of fuzzy inequalities. Here the fuzzy nonlinear programming problem with fuzzy inequalities is considered. Now we quote some definitions of fuzzy set theory.

Definition 1.15 (106). *If X is a collection of objects denoted generically by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:*

$$\tilde{A} = \{(x, \tilde{\mu}_{\tilde{A}}(x)) | x \in X\},$$

$\tilde{\mu}_{\tilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X to the membership space ($M = [0, 1]$). (When M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy and $\tilde{\mu}_{\tilde{A}}(x)$ is identical to the characteristic function of a nonfuzzy set.)

In this thesis we have used the fuzzy inequality due to Zimmermann [103,104].

Definition 1.16. *[Inequality due to Zimmermann [103,104]: This is a fuzzy partial order relation between two real numbers. The fuzzy inequality $a \preceq b$, $a, b \in R$, where $\preceq$ is the fuzzified version of $\leq$, is a fuzzy set whose membership function $\tilde{\mu}(a \preceq b)$ is defined by*

$$\tilde{\mu}(a \preceq b) = \begin{cases} 1 & a \leq b \\ \frac{b+\alpha-a}{\alpha} & b \leq a \leq b+\alpha \\ 0 & a \geq b+\alpha \end{cases}$$

where α is the tolerance limit for the inequality $a \preceq b$ at b.
Similarly the membership function for the inequality $a \succeq b$ can be defined as

$$\tilde{\mu}(a \succeq b) = \begin{cases} 1 & a \geq b \\ \frac{a-b+\beta}{\beta} & b-\beta \leq a \leq b \\ 0 & a \leq b-\beta \end{cases}$$

where β is the tolerance limit for the inequality $a \succeq b$ at b.

Throughout the thesis we have considered Lagrange dual(D_L), Fenchel dual(D_F), Lagrange Fenchel dual(D_{FL}), Wolfe's first order dual(WD), Mond Weir first order dual(MD), Wolfe's second order dual(WD_2), Mond Weir second order dual(MD_2) etc. corresponding to the primal problem (P) as mentioned below.

$$\Big(D_L\Big) \qquad \sup_{q^* \geq 0} \inf_{x \in R^n} [f(x) + q^{*T} g(x)]$$

$$\Big(D_F\Big) \qquad \sup_{p^* \in R^n} [-f^*(p^*) + \inf_{x \in R^n,\ g(x) \leq 0} p^{*T} x]$$

$$\Big(D_{FL}\Big) \qquad \sup_{p^* \in R^n,\ q^* \geq 0} [-f^*(p^*) + \inf_{x \in R^n} [p^{*T} x + q^{*T} g(x)]]$$

$$(MD) \qquad \max_{y,u} \quad f(y)$$
$$\text{subject to} \quad \nabla[f(y) + u^T g(y)] = 0$$
$$u^T g(y) \geq 0,\ u \geq 0.$$

$$(MD_2) \quad \max_{y,u,p} \quad f(y) - \tfrac{1}{2} p^T \nabla^2 f(y) p$$
$$\text{subject to} \quad \nabla[f(y) + u^T g(y)] + \nabla^2 [f(y) + u^T g(y)] p = 0$$
$$u^T g(y) - \tfrac{1}{2} p^T \nabla^2 [u^T g(y)] p \geq 0,\ u \geq 0.$$

$$(WD) \qquad \max_{y,u} \quad f(y) + u^T g(y)$$
$$\text{subject to} \quad \nabla[f(y) + u^T g(y)] = 0,\ u \geq 0$$

$$
\begin{array}{lll}
(WD_2) & \max\limits_{y,u,p} & f(y) + u^T g(y) - \frac{1}{2} p^T \nabla^2 [f(y) + u^T g(y)] p \\
& \text{subject to} & \nabla [f(y) + u^T g(y)] + \nabla^2 [f(y) + u^T g(y)] p = 0
\end{array}
$$

Vector Inequalities:

The following convention for vector relation are used throughout the thesis.

If $x, y \in R^n$, then

$x \geq y \Leftrightarrow x_i \geq y_i$, for $i = 1, 2, \ldots, n$.

$x = y \Leftrightarrow x_i = y_i$, for $i = 1, 2, \ldots, n$.

$x > y \Leftrightarrow x_i > y_i$, for $i = 1, 2, \ldots, n$.

1.3 Literature Survey

In the thesis we have discussed duality results for three types of nonlinear programming problems based on:

1. generalized E-convexity,
2. Fenchel-Lagrange duality,
3. duality for fuzzy nonlinear programming problem.

1.3.1 On generalized E-convexity

In 1999, Youness [99] introduced E-convex function and studied E-convex programming problem. Some of his important results related to $E-$convex programming proved by Youness, are given below.

Theorem 1.3. *Assume that $E(M)$ is convex and $\overline{x}$ is a solution of the following problem:*

$$\begin{array}{lll} (\overline{P}_E) & \min & (f \circ E)(x) \\ & \textit{subject to} & g(x) \leq 0 \\ & & x \in M \qquad , \ M = \{x \in R^n : g_i(x) \leq 0, \ i = 1, \ldots, m\} \end{array}$$

Then, $E(\overline{x})$ is a solution of problem (P).

Theorem 1.4. *Let $E(M)$ be a convex set. If $x^0 = E(z^0) \in E(M)$ is a local minimum of Problem (P) on M, then x^0 is global minimum of problem (P) on M.*

Theorem 1.5. *Assume that $E(M)$ is a convex set and f is strictly E-convex. Then, the solution of Problem $(\overline{P}_E)$ considered in Theorem 1.4, is unique.*

Theorem 1.6. *Let $E(M)$ be a convex set and let $f \circ E, g_i \circ E, i = 1, 2, \ldots, m$, be differentiable on M. Assume that (x^*, y^*) is a solution of the following problem:*

$$\begin{aligned} \nabla[(f \circ E)(x^*) + \langle y^*, (g_i \circ E)(x^*)\rangle] &= 0, \\ \langle y^*, (g_i \circ E)(x^*)\rangle &= 0, \\ (g \circ E)(x^*) \leq 0, \ y &\geq 0. \end{aligned}$$

Then, $E(x^*)$ is an optimal solution of problem (P).

Chen [17] defined semi-E-convex function, quasi-semi-E-convex function, pseudo-semi-E-convex function, logarithmic E-convex function, slack 2-convex set and modified the results of Youness [99] on E-convex programming problem. Some of his important results are given below.

Theorem 1.7. *If $x_0 \in M$ is a fixed point of the mapping $E : R^n \to R^n$ i.e. $x_0 = E(x_0)$ and x_0 is a local minimum of the problem (P) on an E-convex set M, and $f : R^n \to R$ is semi-E-convex on the set M, then x_0 is global minimum of problem (P) on M.*

Theorem 1.8. *If $x_0 \in M$ is a fixed point of the mapping $E : R^n \to R^n$ i.e. $x_0 = E(x_0)$ and x_0 is a local minimum of the problem (P) on an E-convex set M, and $f : R^n \to R$ is pseudo-semi-E-convex on the set M , then x_0 is global minimum of problem (P) on M.*

Theorem 1.9. *Assume function* $f : R^n \to R$ *is a strongly quasi-semi- E-convex on a set* $M \subseteq R^n$*, then the global optimal solutions of problem* (P) *is unique.*

Theorem 1.10. *Let* M *be a nonempty subset of* R^n *,* $E : R^n \to R^n$ *be function, and let* $f : R^n \to R$ *be pseudo-semi-E -convex on E-convex set* $M \subseteq R^n$ *,* $u \in M$ *be fixed point of map E i.e.*$u = E(u)$ *and*

$$\langle f^{'}(E(u)), (E(v) - E(u))\rangle \geq 0, \quad \forall \quad v \in M.$$

Then u *is a minimum of function* f *on* M*.*

Syau and Lee [86] defined E-quasiconvex function, strictly E-quasiconvex function and studied nonlinear programming problem using these generalized E-convex functions. Some of their important results are given below.

Theorem 1.11. *Suppose that* $f : R^n \to R$ *is E-quasiconvex, then the set of solutions of problem* (P) *is convex.*

Theorem 1.12. *Suppose that*
(1) $f : R^n \to R$ *is E-quasiconvex;*
(2) $\overline{x}$ *is a strict local minimizer of* (P)*.*
Then $\overline{x}$ *is also a strict global minimizer of* (P)*.*

Theorem 1.13. *Suppose that*
(1) $f : R^n \to R$ *is E-convex;*
(2) $\overline{x}$ *is a local minimizer of* (P)*.*
Then $\overline{x}$ *is also a global minimizer of* (P)*.*

Theorem 1.14. *Suppose that* $f : R^n \to R$ *is strictly E-quasiconvex.*
(1) If $\overline{x}$ *is a local minimizer of* (P)*, then it is also a global minimizer.*
(2) f *attains it minimum over* $\overline{X}$ *at no more than one point.*

Duca and Lupsa [32] proved some characterizations of E-convex function using a different notion of epigraph. Some of his important results are as follows.

Theorem 1.15. *Let* M *be a nonempty subset of* R^n *and let* $f : M \to R$ *and* $E : R^n \to R^n$ *be two functions. If* M *is an E-convex set and* $epi_E(f)$ *is a convex set, then* f *is an E-convex function on* M*.*

Theorem 1.16. *Let M be a nonempty subset of R^n and let $f : M \to R$ and $E : R^n \to R^n$ be two functions. If M is an E-convex set, $E(M)$ is a convex set, and $epi_E(f)$ is a slack 2-convex set with respect to $E(M) \times R$, then f is an E-convex function on M.*

Theorem 1.17. *Let M be a nonempty subset of R^n and let $f : M \to R$ and $E : R^n \to R^n$ be two functions. Assume that M is an E-convex set and $E(M)$ is a convex set. Then, f is an E-convex function on M if and only if $epi_E(f)$ is a slack 2-convex set with respect to $E(M) \times R$.*

Fulga and Preda [36] defined E-preinvex function and E-prequasiinvex function where differentiation was not required and studied nonlinear programming problem using these generalized E-invex functions.Some of his important results are given below.

Theorem 1.18. *Let $f_j : R^n \to R,\ j = 1, \ldots m$, be E-prequasiinvex functions on R^n; $E(R^n)$ an invex set and the set of feasible solutions nonempty set. Then the set of optimal solutions of the problem is invex.*

Theorem 1.19. *Let $f_0 : R^n \to R$ be a strict E-prequasiinvex function on R^n, $f_j : R^n \to R,\ j = 1, \ldots m$, E-prequasiinvex functions on R^n, $E(R^n)$ an invex set and the set of feasible solutions is a nonempty set. If x^* is a local minimum point of the problem then x^* is a strict global minimum point of the problem.*

All the above authors have defined different types generalized E-convex functions without using differentiability. In Chapter 2, 3, 4, we have defined E-convex function, E-invex, E-quasiinvex function, E-pseudoinvex function, semi-E-invex function, semi-E-quasiinvex function, semi-E-pseudoinvex function etc. with differentiability assumption and studied nonlinear programming problem using these generalized E-convex functions with differentiability. In this respect our result is different from our predecessors.

1.3.2 On Fenchel-Lagrange duality

Lagrange dual (D_L), Fenchel dual (D_F), Fenchel-Lagrange dual (D_{FL}) are constructed using the concept of conjugate function due to Fenchel [34] and Rockfellar [80] in the finite dimensional case. Duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual were studied by Wanka and Bot [89] using convexity. Wanka and Bot [89] have proved the following duality results using convexity assumption.

$$\sup(D_L) \geq \sup(D_{FL}) \quad , \sup(D_F) \geq \sup(D_{FL})$$

$$\inf(P) = \sup(D_L) = \sup(D_F) = \ \sup(D_{FL})$$

All convex functions are not necessarily E-convex on the same domain. If f is an $E-$convex function then it is not necessarily true that $f \circ E$ is a convex function. This motivated us to construct an $E-$convex programming problem for which these duality results can be applicable. We have established a relation between the solution of a general nonlinear programming problem and a general $E-$convex programming problem and studied duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual for $E-$convex programming problem in Chapter 5. In case $E(x) = x$, our duality results reduce to duality results with convexity.

Study of second order dual is important due to its computational advantage over first order dual as its provides tighter bound to the value of the objective function. Second order duality results for Wolfe's dual, Mond Weir dual etc. are studied earlier. But second order duality results for Fenchel dual and Fenchel-Lagrange dual has not been studied yet. In chapter 6, we have constructed second order Lagrange dual, Fenchel dual, Fenchel-Lagrange dual of a general nonlinear programming problem and studied the primal dual relation using convexity and $E-$convexity assumptions.

1.3.3 On fuzzy nonlinear programming problem

Duality for the fuzzy linear programming was first introduced by Rodder and Zimmerman [83] and then further modified by many researchers like Sakawa [84], Wu [97], Zhang and Lee [101], Bector and Chandra [8] etc., in different directions. Also few developments have been made in the area of duality for fuzzy nonlinear programming problems to study the primal dual relationship using different methodologies. Sakawa and Yano [84] proposed a fuzzy dual decomposition method to solve multi objective nonlinear programming problems using Lagrangian multipliers. Zhang, Yuan and Lee [101] used the concept of derivatives for convex fuzzy mapping and developed the necessary and sufficient optimality condition for the existence of solution for fuzzy mathematical programming using Lagrangian dual function which is parallel to KKT condition in crisp case. Recently Wu [97] defined fuzzy valued Lagrange function and proved that the primal and dual fuzzy mathematical programming problems have no duality gap under suitable convexity assumption. But the development in the contest for fuzzy nonlinear programming problem is not enough yet. All the above authors have considered the fuzzy nonlinear programming problem with uncertainties in the form of fuzzy valued function or fuzzy coefficients. They have not used aspiration level for the objective function and constraints in a fuzzy nonlinear programming problem as consider by Bector and Chandra [8] for fuzzy linear programming problem. Bector and Chandra [8] have studied the duality results for fuzzy linear programming problem by converting fuzzy linear programming problem to its crisp equivalent ($(CFLP)$ and $(CFLD)$)which we have discussed in Chapter 7 in detail. Some important results of Bector and Chandra [?] are given below.

Theorem 1.20. *(Modified weak duality). Let (x, λ) be $(CFLP)$-feasible and (w, η) be $(CFLD)$-feasible. Then,*

$$(\lambda - 1)p^T w + (\eta - 1)q^T x \leq (b^T w - c^T x).$$

Theorem 1.21. *Let $(\hat{x}, \hat{\lambda})$ be $(CFLP)$-feasible and $(\hat{w}, \hat{\eta})$ be $(CFLD)$-feasible such that*
$(i)(\hat{\lambda} - 1)p^T \hat{w} + (\hat{\eta} - 1)q^T \hat{x} = (b^T \hat{w} - c^T \hat{x})$,
$(ii)(\hat{\lambda} - 1)p_0 + (\hat{\eta} - 1)q_0 = (c^T \hat{x} - b^T \hat{w}) + (b^T w_0 - c^T x_0)$ and
(iii) the aspiration levels $c^T x_0$ and $b^T w_0$ satisfy $(c^T x_0 - b^T w_0) \leq 0$.
Then $(\hat{x}, \hat{\lambda})$ is $(CFLP)$-optimal and $(\hat{w}, \hat{\eta})$ is $(CFLD)$-optimal.

We have borrowed the idea of Bector and Chandra [8] to study the duality results for fuzzy nonlinear programming problem.

In Chapter 7, we have introduced fuzzy feasible region which is completely a new concept. Using this concept we have studied Lagrange duality results for fuzzy nonlinear programming problem where the objective function and constraint functions are associated with some aspiration levels. In Chapter 8, we have constructed Wolfe's first and second order dual of fuzzy nonlinear programming problem and studied duality results for fuzzy nonlinear programming problem using convexity and E-invexity assumption.

1.4 Important results of the thesis

Optimality and duality results for nonlinear programming problem using convexity and generalized convexity are studied by many researchers during last few decades. In Chapters 2, 3 and 4, we have introduced E-convexity, E-invexity and its generalization for a differentiable function. Also we have studied the optimality and duality results for a nonlinear programming problem and its dual using E-invexity and generalized E-invexity assumption.

Chapter 2

In this chapter differentiable E-convex and semi-E-convex functions are introduced. The sufficient optimality condition for the existence of local optimum of a nonlinear programming problem under semi-E-convexity is derived and the results are verified through numerical examples. Some of the important results of this chapter are given below.

Theorem 1.22. *Let $M \subseteq R^n$, $E : R^n \to R^n$ be a homeomorphism. If $f : M \to R$ has a local minimum point in the neighborhood of $E(\overline{x})$, then f is E-convex at $\overline{x}$.*

Theorem 1.23. *Let M be a nonempty open E-convex subset of R^n, $f : M \to R$ and $E : R^n \to R^n$ are differentiable functions. Let E be a homeomorphism. Then f is E-convex at $\overline{x} \in M$ if and only if*

$$(f \circ E)(x) \geq (f \circ E)(\overline{x}) + (\nabla(f \circ E)(\overline{x}))^T(E(x) - E(\overline{x}))$$

for all $E(x) \in N_\epsilon(E(\overline{x}))$ where $N_\epsilon(E(\overline{x}))$ is ϵ-neighborhood of $E(\overline{x})$, $\epsilon > 0$.

Theorem 1.24. *Let M be a nonempty open E-convex subset of R^n, $f : M \to R$ and $E : R^n \to R^n$ are differentiable functions. Let E be a homeomorphism and $\overline{x}$ be a fixed point of E. Then f is semi-E-convex at $\overline{x} \in M$ if and only if*

$$f(x) \geq f(\overline{x}) + (\nabla(f \circ E)(\overline{x}))^T(E(x) - E(\overline{x}))$$

for all $E(x) \in N_\epsilon(E(\overline{x}))$ where $N_\epsilon(E(\overline{x}))$ is ϵ-neighborhood of $E(\overline{x})$, $\epsilon > 0$.

Theorem 1.25. *(Sufficient optimality condition) Let M be a nonempty open E-convex subset of R^n, $f : M \to R$, $g : M \to R^m$, and $E : R^n \to R^n$ are differentiable functions. Let E be a homeomorphism and $\overline{x}$ be a fixed point of E. Let f and g are semi-E-convex at $\overline{x} \in M'$. If $(\overline{x}, \overline{y}) \in M' \times R^m$ satisfies the following system*

$$\nabla[(f \circ E)(x) + \langle y, (g \circ E)(x)\rangle] = 0,$$
$$\langle y, (g \circ E)(x)\rangle = 0, y \geq 0,$$

then $\overline{x}$ is local optimal solution of (P).

Chapter 3

In this chapter generalized differentiable E-invex, E-quasiinvex, E-pseudoinvex, semi-E-invex, semi-E-quasiinvex, semi-E-pseudoinvex functions are introduced. The sufficient optimality condition and the duality results for the nonlinear programming problem under generalized E-invexity are derived and the results are verified through examples. Some of the important results of this chapter are given below.

Theorem 1.26. *(Sufficient optimality condition) Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $E(M)$ be an open set in R^n. Suppose $f : M \to R$, $g : M \to R^m$ and E are differentiable functions on M. If f is E-pseudoinvex function and for $u \geq 0$, $u^T g$ is semi-E-quasiinvex function with respect to same η at $x \in M'$, where $M' = \{x \in M |\ g(x) \leq 0\}$, x is a fixed point of the map E, $(x, u) \in M' \times R^m$, $u \geq 0$ satisfy the following system*

$$\nabla[(f \circ E)(x) + \langle u, (g \circ E)(x)\rangle] = 0,$$
$$\langle u, (g \circ E)(x)\rangle = 0,$$

then x is a local optimal solution of (P).

Theorem 1.27. *Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $E(M)$ be an open set in R^n. Suppose $f : M \to R$, $g : M \to R^m$ and E are differentiable functions on M. Let x be a feasible solution of (P) and (y, u) be feasible solution of (MD), where $y \in M$*

is a fixed point of the map E. If f is E-pseudoinvex function with respect to η and $u^T g$ is semi-E-quasiinvex function with respect to same η at y then

$$f(E(x)) \geq f(E(y)) \text{ for all } x \in M'.$$

Theorem 1.28. *Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $E(M)$ be an open set in R^n. Suppose $f : M \to R$, $g : M \to R^m$ and E are differentiable functions on M. If x is an optimal solution of (P), where x is a fixed point of the map E at which KKT constraint qualification is satisfied then there exists $u \in R^m$, $u \geq 0$ such that (x, u) is a feasible solution for (MD). If f is E-pseudoinvex function with respect to η on M and $u^T g$ is semi-E-quasiinvex function with respect to same η at $x \in M$, then the optimal values of (P) and (MD) are equal on $E(M)$.*

Chapter 4

This chapter is a continuation of Chapter 3. Here the notion of second order generalized differentiable E-invex, E-quasiinvex, E-pseudoinvex, semi-E-invex, semi-E-quasiinvex, semi-E-pseudoinvex functions are introduced. The duality results for the nonlinear programming problem under second order generalized E-invexity are derived. Some of the important results of this chapter are given below.

Theorem 1.29. *(Weak Duality) Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $E(M)$ be an open set in R^n. Suppose $f : M \to R$, $g : M \to R^m$ and E are differentiable functions on M. Let x be a feasible solution of (P) and (y, u, p) be feasible solution of (MD_2), where $y \in M$ is a fixed point of the map E. If f is second order semi-E-pseudoinvex function with respect to η and $u^T g$ is second order semi-E-quasiinvex function with respect to same η at y then*

$$f(x) \geq f(y) - \frac{1}{2} p^T \nabla^2 f(y) p \text{ for all } x \in M'.$$

Theorem 1.30. *(Strong Duality) Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $E(M)$ be an open set in R^n. Suppose $f : M \to R$, $g : M \to R^m$ and E are differentiable functions on M. If x is an optimal solution of (P), where x is a fixed point of the* *map E at which KKT constraint qualification is satisfied, then there exists $u \in R^m$, $u \geq 0$ such that $(x, u, p = 0)$ is feasible solution for (MD_2). If f is second order semi-E-pseudoinvex function with respect to η on M and $u^T g$ is second order semi-E-quasiinvex function with respect to same η at $x \in M$ then the optimal values of (P) and (MD_2) are equal.*

Duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual were studied by Wanka and Bot [89] using convexity. In Chapters 5 and 6, we have studied the duality results for Lagrange dual, Fenchel dual and Fenchel-Lagrange dual using convexity and E-convexity assumptions.

Chapter 5

In this chapter strong duality results between a primal nonlinear programming problem and its Lagrange dual, Fenchel dual and Fenchel-Lagrange dual are proved using E-convex functions. Some of the important results of this chapter are given below.

Theorem 1.31. *Let M be a nonempty subset of R^n, M is an E-convex set and $E(M) = M$. If $f : M \to R$, $g : M \to R^m$ are E-convex functions and $f(E(x)) \geq f(x)$, $g(E(x)) \geq f(x)$ $\forall$ $x \in M$, then* $\sup(D_L^E) = \sup(D_{FL}^E)$.

Theorem 1.32. *Let M be a nonempty E-convex subset of R^n, $E : R^n \to R^n$ such that $E(M) = M$. If g is E-convex function on M , $G = \{x \in M : (g \circ E)(x) \leq 0\} \neq \emptyset$ and (CQ) is satisfied then* $\sup(D_F^E)= \sup(D_{FL}^E)$.

Theorem 1.33. *(Strong duality) Let M be a nonempty E-convex subset of R^n, $f : R^n \to R$ and $g : R^n \to R^m$, are E-convex functions on M, $E : R^n \to R^n$ such that $G = \{x \in M : (g \circ E)(x) \leq 0\} \neq \emptyset$, (CQ) is satisfied. If* $\inf(P_E)$ *is finite then strong duality holds that is*

$$\inf(P_E) = \sup(D_L^E)$$

Theorem 1.34. *(Strong duality) Let M be a nonempty E-convex subset of R^n, $E : R^n \to R^n$ such that $E(M) = M$. If*
(i)f and g are E-convex functions on M,

(ii) $G = \{x \in M : \ (g \circ E)(x) \leq 0\} \neq \emptyset$ *and (CQ) is satisfied,*
(iii) $\inf(P_E)$ *is finite,*
(iv) $(f \circ E)(x) \geq f(x)$, $(g \circ E)(x) \geq g(x) \ \forall \ x \in M$
then

$$\inf(P_E) = \sup(D_L^E) \ = \sup(D_F^E) = \ \sup(D_{FL}^E).$$

Chapter 6

In this chapter second order Lagrange dual, Fenchel dual and Fenchel-Lagrange dual for a nonlinear programming problem are constructed. Weak duality results are proved and duality relationship between these three second order duals under appropriate assumptions are established. Some of the important results of this chapter are given below.

Theorem 1.35. *If* f *and* $g_i, i = 1, \ldots, m$ *are convex functions then*

$$\sup(D_L^2) = \ \sup(D_{FL}^2).$$

Theorem 1.36. *If* $g_i, i = 1, \ldots, m$ *are convex functions such that* $G = \{(x, \ r) : x, \ r \in R^n, \ G_2(x, r) \leq 0\} \neq \emptyset$ *and* $(\bar{C}Q)$ *is satisfied then*

$$\sup(D_F^2) = \sup(D_{FL}^2).$$

Theorem 1.37. *(Strong duality) If* $f : R^n \to R, \ g_i : R^n \to R, i = 1, \ldots, m$ *are convex functions such that* $G = \{(x, \ r) : x, \ r \in R^n, \ G_2(x, r) \leq 0\} \neq \emptyset$, $(\bar{C}Q)$ *is satisfied and* $\inf(P)$ *is finite then*

$$\inf(P^2) = \sup(D_L^2) = \sup(D_F^2) = \ \sup(D_{FL}^2).$$

Theorem 1.38. *If* f *and* $g_i, i = 1, \ldots, m$ *are* $E-$*convex functions on* R^n *and* $E : R^n \to R^n$ *is a linear function then* $\sup(D_L^{E2}) = \sup(D_{FL}^{E2})$. *In addition to this if* $\{(x, \ r) : x, \ r \in R^n, \ {G_2}^E(x, r) \leq 0\} \neq \emptyset$ *and there exists at least one* $(x', r') \in R^n \times R^n$ *such that* ${G_2}^E(x', r') < 0$, *then* $\sup(D_F^{E2}) = \sup(D_{FL}^{E2}) = \sup(D_L^{E2}) = \inf(P_E^2)$.

Bector and Chandra [8] have studied duality results for fuzzy linear programming problem. In Chapters 7 and 8, we have studied duality results for fuzzy nonlinear programming problem.

Chapter 7

This chapter deals with duality results of nonlinear programming problem in fuzzy scenario. Here the concept of fuzzy feasible region is introduced. Also crisp equivalent of Lagrange dual for fuzzy primal nonlinear programming problem is formulated. The weak and strong duality relationship between them are studied and verified through an example. The following results are proved in this chapter.

Theorem 1.39. *For every feasible solution of* (FNP) *there exists* $\lambda \in [0,1]$ *such that* (x, λ) *is a feasible solution of* $(CFNP)$.

Theorem 1.40. *Let* x *be a feasible solution of* (FNP) *and (u,y) be a feasible solution of* (FND) *then there exists* $\lambda, \eta \in [0,1]$ *such that*

$$(\lambda - 1)p_0 + (\eta - 1)q_0 + W_0 - Z_0 \leq L(u, y) - f(x).$$

In particular weak duality holds if $x \in FR(p_0^{'}, 0)$, *for any* $p_0^{'} \in [0, p_0]$ *and strong duality holds if* $x \in FR(p_0^{'}, p^{'}) \setminus FR(p_0^{'}, 0)$, *for any* $p^{'} \in [0, p]$.

Chapter 8

This chapter is a continuation of Chapter 7. In this chapter crisp equivalent of Wolfe's first order dual and Wolfe's second order dual for fuzzy primal nonlinear programming problem are formulated. The weak and strong duality relationship between them are studied and verified through an example. The following results are proved in this chapter.

Theorem 1.41. *Let* (x_0, λ_0) *be a feasible solution of* $(CFNP)$ *and* (x^*, u^*, η^*) *is a feasible solution of* $(CFWD)$. *If* f *and* g *are differentiable convex functions at* x^* *then*

$$(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u^{*T}p + (\eta^* - 1)q_0 \leq Z_0 - W_0.$$

Theorem 1.42. *Let* (x_0, λ_0) *be a feasible solution of* $(CFNP)$ *and* (x^*, u^*, s^*, η^*) *be a feasible solution of* $(CFWD_2)$*. If* f *and* g *are twice differentiable convex functions at* x^* *then*

$$(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u^{*T}p + (\eta^* - 1)q_0 \leq Z_0 - W_0 \ for \ s^* = (x_0 - x^*) \ .$$

Theorem 1.43. *Suppose*

1. $E : R^n \to R^n$ *is a differentiable function,* $E(R^n)$ *is an open set of* R^n *and* x^* *is a fixed point of* E*,*

2. f *and* g *are differentiable semi-E-invex functions at* x^* *with respect to* η*, where* $\eta : R^n \times R^n \to R^n$*,*

3. (x_0, λ_0) *and* (x^*, u^*, η^*) *are feasible solutions of* $(CFNP)$ *and* $(CFWD)$ *respectively,*

then $(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u^{*T}p + (\eta^* - 1)q_0 \leq Z_0 - W_0$.

Theorem 1.44. *Suppose*

1. $E : R^n \to R^n$ *is a twice differentiable function,* $E(R^n)$ *is an open set of* R^n *and* x^* *is a fixed point of* E*,*

2. f *and* g *are twice differentiable semi-E-invex functions at* x^* *with respect to* η*, where* $\eta : R^n \times R^n \to R^n$*,*

3. (x_0, λ_0) *and* (x^*, u^*, s^*, η^*) *are feasible solutions of* $(CFNP)$ *and* $(CFWD_2)$ *respectively, then*

$(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u^{*T}p + (\eta^* - 1)q_0 \leq Z_0 - W_0$.

Chapter 9

This is the concluding chapter which includes some important remarks and future plan of our research work.

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