Chapter - 1

Introduction

1.1 Motivation

In this work an attempt is made to create an integrated approach to a class of systems with random excitations, both external (additive) and parametric (multiplicative). Usual approaches to stochastic systems start with mathematical models of the underlying systems on which the stochastical natures of existing signals and parameters are grafted. The morphological aspects of the physical or abstract systems remain entirely obscured. There grows a wide gulf between the system and it's stochastical formulation. In the present work, the dynamics of stochastical systems is approached keeping the model of the basic system in the foreground. The bondgraph representation of a system not only assists in unambiguous representation of system connectivities, it also provides causal orientation to the system model. The notion of causality, through bondgraphs, has occupied the central stage in system modelling. Well developed semantics of causality of the bondgraph theory provide the modeller with an over all structure of the mathematical model even before the equations (algebraic and differential) are arrived at. The modeller gets his conceptualization of the system, ratified by revealing the model's physical admissibility and mathematical solvability. After addressing the issue of stochastical systems' modelling, this work proceeds to provide a structured approach to create mathematical models to predict stochastical moments for a class of systems with polynomial non-linearities. The ideas and actions to bring this program to a satisfactory end, are fairly interlaced. Both actions adapted and alternatives, and the supporting ideas are presented in *Fig. 1.1.* From this one may get an impression of the arena in which the present work is placed.

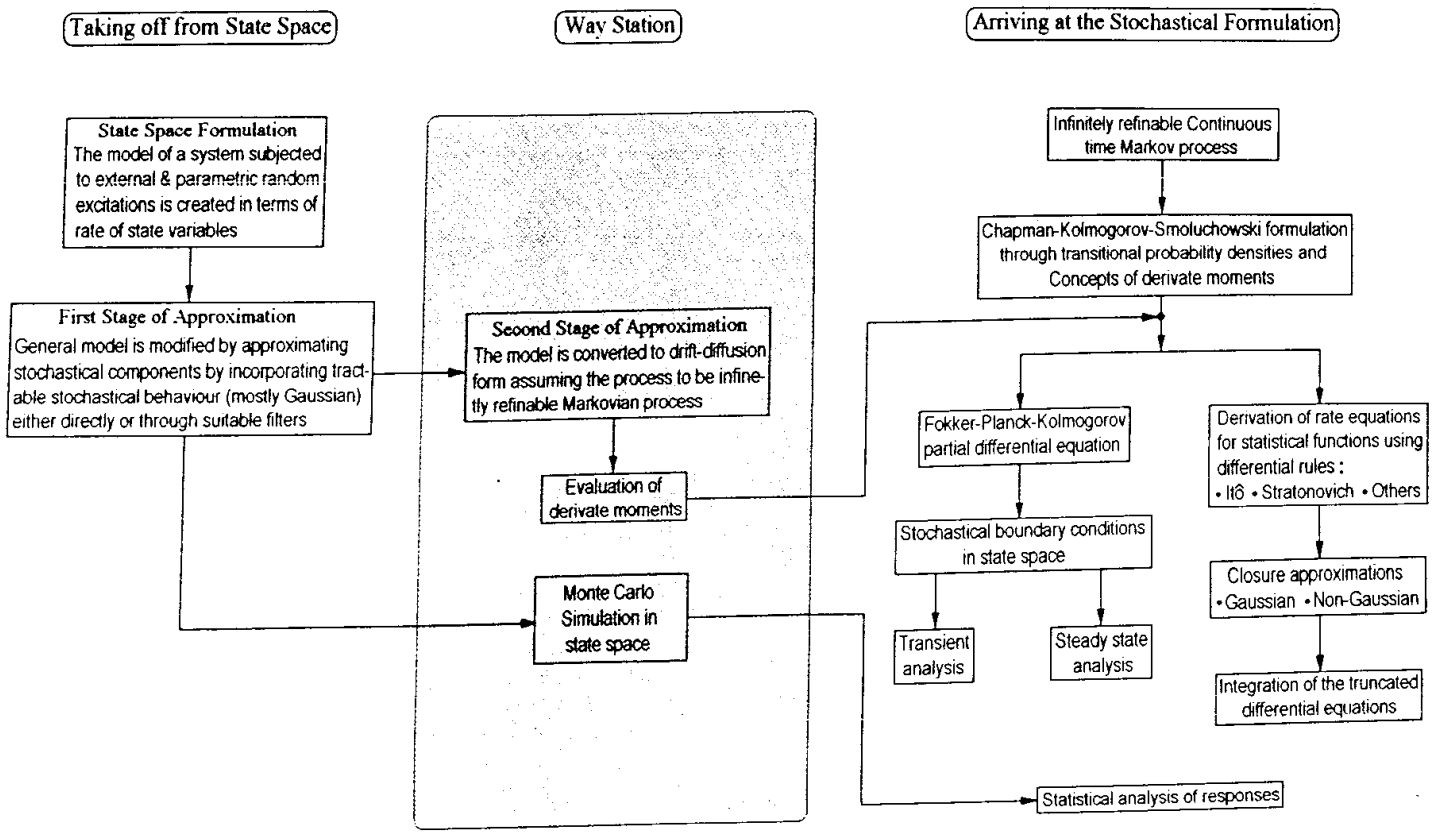

Fig. 1.1 A route from state space models to stochastical formulations

1.2 A Route from State Space Models to Stochastical Formulations

1.2.1 State space formulation

Modelling of systems residing in multi-energy domain and arriving at the system equations is not a trivial task. A system can, nearly always, be conceptually dichotomed in two sets, one of which consists of energy storers, dissipators and importers and the other signifying the constraint structure. This constraint structure neither creates nor destroys the power. In 1959, Prof. H. M. Paynter created a set of powerful symbolisms which expresses this deep, though abstract interplay of power and energy in a concrete and discernible form, in a diagram which gives firm grasp of essential dynamics of the system and aids creation of motion equations and prediction. This power exchange portrayal of a system is called **Bondgraph**[86].

Bondgraphs are independent formulation of classical system dynamics which reveal many obscured structures in it, unify it and most significantly stimulate the imagination of the modeller by manifestation of concrete and abstract ideas in integrated forms. Further, this modelling language provides a strict law of causation, that identifies the cause and the effect of the dynamical quantities and thus maintains the philosophy that the history function should always be integrated. Any violation of the laws of causality in a system model hampers the system admissibility. Thus causally augmenting a bondgraph can be used as an analysis tool in order to get an a-priori idea regarding the validity of the system model, the state vectors of the model, and the ways in which the system design may be extended to achieve the desired performance.

One of the major objectives behind causal augmentation of the bondgraph, besides the analysis of the appropriateness, is to obtain a set of assignment statements suitable for simulation. Bondgraphs lead to generation of a set of first order differential equations. Derivation of these equations can be done using several software, though the apparent form of the output may vary. Two such popular software are *COSMO* [125] and *SYMBOLS* [1].

Bondgraph technique could be used for the modelling of stochastical processes. The equations of motion of stochastical process obtained from a bondgraph model could be written in the following form

$$\frac{d}{dt}\bar{X}(t) = \bar{f}(t, \bar{X}(t)) + \left[\sigma(t, \bar{X}(t))\right] \bar{V}(t), \tag{1.1}$$

where $\bar{X}(t)$ is a n-dimensional state vector of the response coordinates, $\bar{f}(t, \bar{X}(t))$ is a vector function of the state variables, $[\sigma(t, \bar{X}(t))]$ is an $n \times \mu_s$ matrix function, and $\bar{V}(t)$ is a μ_s-dimensional independent random processes which influence the model through matrix $[\sigma(t, \bar{X}(t))]$. Equation (1.1) is known as *Langevin* equation. The vector $\bar{V}(t)$ is a vector of *white noises*. The 'white noise' is conceived as a random process with mean value zero and a constant spectral density on the entire real axis. Such a process does not exist in the conventional sense, since it would have to have the Dirac delta function as covariance, and hence an infinite variance (and independent values at all points). Nonetheless, the 'white noise' is a useful mathematical idealization for describing random influences that fluctuate rapidly and hence are virtually uncorrelated for different instants at time.

1.2.2 First stage of approximation

Markov processes

In the past few decades, considerable effort has been expended towards developing techniques to predict the response of non-linear stochastical systems subjected to external and parametric random excitations. General class of stochastical processes defy both formulation and mathematical approach. A widely accepted approximation is that the processes are assumed to be Markovian, or in the context of continuous time systems, infinitely refinable Markov process. The groundwork for the theory of Markov processes was laid in 1906 by A. A. Markov.

A stochastic dynamical system is said to be Markovian if the probable (future) state of the system at any time $t > \tau$ is independent of the (past) behaviour of the system at times $t < \tau$, given the present state at time τ.

For instance, a continuously valued stochastic process $X(t)$ on $[0,T]$ will be a Markov process if for $n = 1, 2, 3, \cdots$ and any sequences $0 \leq t_0 < \cdots < t_n \leq T$ and $x_0, x_1, \cdots, x_n$, the following relation exists:

$$\begin{aligned} \Pr\left(X(t_n) < x_n | X(t_{n-1}) = x_{n-1}; X(t_{n-2}) = x_{n-2}; \cdots; X(t_0) = x_0\right) \\ = \Pr\left(X(t_n) < x_n | X(t_{n-1}) = x_{n-1}\right). \end{aligned} \tag{1.2}$$

The relation(1.2) means that probability of $X(t_n) < x_n$ depends on values acquired at various time points before the time t_n. The Markovian process is a subclass of this general process which depends entirely on single value acquired by the process in past. In this sense, Markov process is an one step memory random process.

The importance of the Markov process resides in the fact that when the excitation of a linear or non-linear dynamic system is ideal white noise, the response is a Markov process. In such a case transition probability density function of the process is governed by a partial differential equation, popularly known as Fokker-Plank-Kolmogorov equation. The theory of Markov process was first applied to linear systems ([31], [97], [146], [147]) to calculate the probability density function of system response to Gaussian white noise or shot noise excitation. Recently, Lin ([100], [103]) has justified the approximation of a non-linear stochastical system as a Markov process. The theory of Markov processes is elaborately discussed in the literature, for example Doob [48], Bharucha-Reid [15].

Diffusion processes

A Markov process $X(t)$ on $[0,T]$ is called a diffusion process if its transition probability satisfies the following two conditions:

(i) For every $\epsilon > 0$, t, and $X(t)$,

$$\lim_{dt \to 0} \frac{1}{dt} \int_{|y-x|>\epsilon} p(X(t+dt) = y | X(t) = x) dy = 0. \tag{1.3}$$

(ii) There exist functions $f(t, X(t))$ and $\sigma(t, X(t))$ such that for all $\epsilon > 0$, t, and $X(t)$,

$$\lim_{dt\to 0}\frac{1}{dt}\int_{-\infty}^{\infty}(y-x)p(X(t+dt)=y|X(t)=x)dy = f(t,X(t))$$
$$\lim_{dt\to 0}\frac{1}{dt}\int_{-\infty}^{\infty}(y-x)^2p(X(t+dt)=y|X(t)=x)dy = \sigma^2(t,X(t)). \quad (1.4)$$

The function $f(t,X(t))$ is called the (*infinitesimal*) *drift coefficient* of $X(t)$ and $\sigma^2(t,X(t))$ is called the (*infinitesimal*) *diffusion coefficient.*

Gaussian and Poisson processes

A stochastic process $\{X(t), t \in [0,T]\}$ is a *Gaussian* or *normal* process if the probability distributions of all orders are completely determined by knowledge of the mean value for all times t and knowledge of the correlation function for all pairs of times t_1 and t_2. The white noise process $V(t)$ in eq.(1.1) is usually idealized as a *Gaussian* process. The engineering importance of a *Gaussian* process stems from the fact that when a large number of small independent random effects are superimposed, then, regardless of their individual distribution, the distribution of the sum of these effects is approximately Gaussian.

Poisson processes belong to the general class of counting processes which arise in problems concerning counting the natural numbers. A random process is said to be a *Poisson* process if, (1) the arrivals in the future are independent of the arrival in the past, (2) arrival rates are stationary, and (3) probability of simultaneous arrivals are negligible.

Wiener processes

A stochastic process that is of great importance in theory and applications is the *Brownian motion* or *Wiener* or *Wiener-Levy* process. A continuous parameter process $X(t)$ on $[0,T]$ is a Brownian motion process if

- $\{X(t), t \geq 0\}$ has stationary independent increments;
- for every $t \geq 0$, $X(t)$ is normally distributed;
- for every $t \geq 0$, $E[X(t)] = 0$;

- $\Pr\{X(0) = 0\} = 1$.

Stochastic differential equations

For strict application of the Markov vector approach, the white noise process $V_i(t)$ in eq.(1.1) must be regarded as mathematical white noise having a constant spectral density up to infinite frequency. The process become physically unrealizable as the mean square reaches infinity and correlation function become dirac delta function [85], [48], [60]. Consequently, eq.(1.1) has no mathematical meaning. The mathematical difficulty has been resolved by using an alternative approach.

The white Gaussian noise $\bar{V}(t)$, can be formally written as the derivative of Wiener process $\bar{W}(t)$,

$$\bar{V}(t) = \frac{d}{dt}\bar{W}(t). \tag{1.5}$$

As a result of this definition, eq.(1.1) can be written in the form of the differential equation

$$d\bar{X}(t) = \bar{f}(t, \bar{X}(t))\, dt + \left[\sigma(t, \bar{X}(t))\right]\, d\bar{W}(t). \tag{1.6}$$

Equation (1.6) is the well known Itô stochastic differential equation. Here, incremental properties of the Wiener process are :

$$E[d\bar{W}(t)] = 0,$$
$$E[d\bar{W}(t)d\bar{W}^t(t)] = I dt,$$

where I is the identity matrix. The process $\bar{X}(t)$ described by the Itô differential eq.(1.6) is also known as Itô process and the study of the properties of this process is defined as *Itô calculus* [74]. Equation (1.6) is meaningful only if its integral is defined:

$$\bar{X}(t) - \bar{X}(t_0) = \int_{t_0}^{t} \bar{f}(\tau, \bar{X}(\tau))\, d\tau + \int_{t_0}^{t} \sigma(\tau, \bar{X}(\tau))\, d\bar{W}(\tau). \tag{1.7}$$

The first integral can be defined as an mean-square Riemann integral or as an ordinary integral for the sample functions. The second integral has not

yet been defined. Because of the erratic properties of the Wiener pròcess, it cannot be defined for the sample functions. This integral is specifically defined as Itô stochastic integral. Doob ([48]) and Kushner ([92], [93]) has discussed the properties and the calculus of operations involving stochastic integrals.

Stratonovich [136] has proposed a new definition of a stochastic integral. The equivalent Stratonovich equation of eq.(1.6) is

$$d\bar{X}(t) = \left[\bar{f}(t,\bar{X}(t)) - \frac{1}{2}\sigma(t,\bar{X}(t))\frac{\partial\sigma(t,\bar{X}(t))}{\partial\bar{X}}\right] dt + \sigma(t,\bar{X}(t))\, d\bar{W}(t). \quad (1.8)$$

1.2.3 Monte Carlo simulations

A method for estimating the response statistics of randomly excited nonlinear stochastical systems, within any desired confidence level, is based on random computational experiments, popularly known as Monte Carlo simulations. For a Monte Carlo simulation, the stochastic model eq.(1.6) is written as,

$$d\bar{X}(t) = \bar{f}(t,\bar{X}(t))\, dt + \left[\sigma(t,\bar{X}(t))\right]\ \bar{\mathcal{R}}(t), \quad (1.9)$$

where $\bar{\mathcal{R}}(t) = d\bar{W}(t)$. The elements of $\bar{\mathcal{R}}(t)$ are *Gaussian* random numbers. Monte Carlo method consists of generating a large number (n) of sample excitations, and processing them to obtain the desired response statistics. In the past, both analog ([40], [76], [105]) and digital ([129], [53]) computational systems have been used for Monte Carlo simulations.

Though the Monte Carlo method is applicable to both stationary and nonstationary response of systems, the main drawback of this method is the computational cost. The numbers of sample records which are necessary for the estimation of the response statistics, within commonly acceptable levels, is of the order of five hundred. To gain one additional significant figure in a result requires a hundredfold increase in computational cost. Spanos [134] has estimated that, for cases where both equivalent linearization and numerical simulation can be applied, the computational efficiency of the former will be order of 100 to 1000 times better than the later. When exact solution is