0. Introduction

0.0. Background

It is very well known that every plane figure reciprocates into its dual plane figure and every solid figure into its dual solid figure. In the first case reciprocation is performed wo (with respect to) a conic in a projective plane and in the second case wo a quadric in a projectire space of 3 dimension S_3. Every point in a plane has a polar line and every line a pole therein, Every point in S_3 has a polar plane and every plane a pole wo a quadric therein. Every figure f as a locus of points reciprocates into an envelope f' of the polars of its points, and f' back into f. The two figures f and f' are therefore called reciprocal of each other, and their reciprocity is mutual. This method of reciprocation or polarity was first discovered by 'a remarkable Frenchman, Jean Victor Poncelet (1788-1867), who fought in Napoleon's Russian campaign (1812) until the Russian took him prisoner. Being deprived of all books, he decided to reconstruct the whole science of geometry. The result was his epochmaking Traite des proprietes projectives des figure which was first published in 1822' (Coxeter [1], p.4). We can define polarity wo higher plane curves (Salmon [1], p.50) as well as wo surface of higher order in space (Mandan [5], Room, p.136; Salmon [2], pp.179-180; Semple and Roth, pp.10-11; Todd, pp.104-107). Polarity is also well defined wo a triangle (Court [1], pp.244-256; Coxeter, [1] p.113; Todd, pp.74-76) considered as a degenerate plane cubic, as well as wo a tetrahedron (Court [2], pp.266-267) taken as a degenerate quartic surface. But the triangle formed by the polar lines of the vertices of a triangle and one formed of the poles of its sides wo a given triangle are never same. In other words there exist no reciprocal triangles wo a triangle.

There arise then a problem whether there exist tetrahedra reciprocal to each other wo a tetrahedron. The answer to this query forms the subject matter of this thesis.

0.1. Projective Coordinates

We use here the algebraic symbol of every point X by the same letter as used earlier by Geometers like Baker ([1], pp.69-74; [2], pp.115-160), Coxeter ([2]), Mandan ([1], [3], [4]) and Room (p.5) to a great advantage as a synthesis of the two tools of pure and algebraic geometries.

In S_3, any 5 points A_i ($i = 0, 1, 2, 3, 4$), no four being coplanar, are always connected by a <u>syzygy</u> which by proper choice of symbols may be expressed as

$$(0.1.0) \qquad A_i = 0$$

Therefore no further multiplication of these symbols by any algebraic symbol is legitimate, save by one the same for all. We suppose that the 5 points not be connected by any further <u>syzygy</u> .

The <u>syzygy</u> shows that we may take $T(A) = A_0A_1A_2A_3$ as the tetrahedron of reference, A_4 as its <u>unit point</u>, and the symbol for any other point X in S_3 as

$$(0.1.1) \qquad X = \sum_{i=0}^{3} x_i A_i, \qquad x_i \in F$$

where F is the field of complex numbers.

DEFINITION - 0.1. The 4 numbers x_i are then defined as the <u>projective</u> or <u>homogeneous coordinates</u> of the point X wo $T(A)$ and its unit point A_4 and written as a row or 1 x 4 <u>matrix</u>

$$(0.1.2) \qquad X = (x_0, x_1, x_2, x_3)$$

determined uniquely except for a common multiple. That is, $kX = (kx_0, kx_1, kx_2, kx_3)$, $k \in F$, represents the same point X (Seidenberg, pp.141-143). The vertices of $T(A)$ are then represented algebraically by the 4 rows of the 4 x 4 identity matrix

$$(0.1.3) \quad \left[T(A)\right] = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the point X by the product of the preceding two matrices as

$$(0.1.4) \qquad X = X\left[T(A)\right]$$

which explains the identity of its two algebraic syblols (0.1.1) and (0.1.2).

0.2. Linear Transformation.

Let $T(B)$ be a tetrahedron other than $T(A)$ such that its vertices are represented algebraically by the 4 rows of the 4×4 matrix

$$(0.2.0) \quad \left[T(B)\right] = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{12} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$= \left[T(B)\right]\left[T(A)\right]$$

wo the tetrahedron $T(A)$ or

$$(0.2.1) \qquad B_i = (b_{i0}, b_{i1}, b_{i2}, b_{i3})$$

Now every point X can be refered to $T(A)$ as well as to $T(B)$ and therefore represented algebraically as

$$(0.2.2) \qquad X[T(A)] = \sum_{i=0}^{3} x_i A_i = \sum_{i=0}^{3} x_i' B_i = X'[T(B)]$$

or

$$(0.2.3) \qquad X'[T(B)] = X[T(A)] = X[T(B)]^{-1}[T(B)]$$

$[T(A)]$ being the identity matrix (0.1.3) and $[T(B)]^{-1}$ the inverse matrix of $[T(B)]$. Therefore we have

$$(0.2.4) \qquad (x_0', x_1', x_2', x_3') = (x_0, x_1, x_2, x_3)[T(B)]^{-1}$$

$$= \frac{k}{|T(B)|}(x_0, x_1, x_2, x_3)\begin{bmatrix} B_{00} & B_{10} & B_{20} & B_{30} \\ B_{01} & B_{11} & B_{21} & B_{31} \\ B_{02} & B_{12} & B_{22} & B_{32} \\ B_{03} & B_{13} & B_{23} & B_{33} \end{bmatrix}$$

where $k\,B_{ij}$ is the cofactor of the elements b_{ij} in the determinant $|T(B)|$ of the matrix $[T(B)]$.