Chapter 1

Introduction

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1.1 Inventory control

The concept of *Operations Research* owes its origin to the military administration during World War II. The military management gathered together scientists from various disciplines to develop ways for the integrative and efficient execution of various military projects and plan for various battlefield situations. This concept injected a multidisciplinary and systematic approach to the operations of different systems, putting it in the framework of scientific formalism, and thus came to be known as *Operations Research*. Wartime concepts and technologies often gain prominence in non-military applications in the post-war period. Following this trend, the *Operations Research* concept also attracted the attention of managers worldwide for developing solutions to their complex real life problems. Since then, *Operations Research* has been playing a pivotal role and has brought about a paradigm shift in decision making problems of various areas like Business Management, Administration, Economics, Behavioral Sciences, and Engineering etc. *Operations Research* is mainly concerned with the application of scientific knowledge and techniques to real life decision making problems. Its main aim is to provide the expert / manager with a better understanding of the problem and with a structured view of the logistics involved so as to enable him to come up with better solutions for them. In the functional areas of management, one of the principal fields of focus in *Operations Research* is Inventory Control.

In business management, one of the most important decisions to be taken is the procurement and maintenance of resources. When the resource involved is materials or goods that are usable but at present idle, it is commonly referred to as stock or inventory. So the word 'inventory' usually refers to the stock of goods that is maintained to ensure the smooth and efficient running of business. Now the stocking of goods is dependent on various factors such as demand, time-lag between placement of orders and their receipt, deterioration, seasonal fluctuations etc. The study of these types of problems regarding the procurement and maintenance of stock is known as Inventory Management or Inventory Control.

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Some of the reasons why Inventory Control is so essential to all kinds of business are given below:

(*i*) It ensures an immediate and adequate supply of items to the customers. This prevents the need for frequent procurements and thereby helps reduce the overall operational investment.

(*ii*) It serves as buffer stock between the provider and the customer in case of delayed arrivals of orders, frequent rejection etc. This helps improve the goodwill of the provider as the customers do not have wait to receive their orders.

(*iii*) The price of a product may be subject to seasonal fluctuations. In such cases it is profitable for the manufacturer / supplier to stock sufficient quantity of the product when its price is low.

(*iv*) It helps in reducing the loss due to deterioration, obsolescence or damage etc. of products.

(*v*) Sometimes a price discount is offered for bulk purchasing. Maintaining sufficient inventory is essential for the manufacturer / supplier to provide such offers.

(*vi*) It reduces the possibility of duplicating orders. The customer can choose from a large pile of products. A large pile of goods displayed also attracts the customer to purchase more as in the case of supermarkets.

(*vii*) It helps reduce the cost of the product to the customer by helping reduce the overall procurement / manufacturing expenditure. It also improves the utilization of manpower, equipment and facility and thereby ensures a more smooth and efficient running of business.

However, though the maintenance of inventories is essential due to the reasons listed above, it also means the locking up of capital in terms of investment as maintaining inventories incurs costs associated with rent for storage space, equipments, personnel, insurance etc. Thus simply maintaining an excess of inventory is not a very economically desirable decision. So an optimum level of inventory needs to be maintained to ensure the smooth and uninterrupted supply of goods while at the same time the process needs to be economically viable. This trade-off between maintaining inventory and the cost of

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maintaining it leads to the fundamental questions of Inventory Control which are as follows:

(*i*) What quantity of an item is to be ordered / procured / manufactured for replenishment i.e., what is the optimal replenishment quantity?

(*ii*) When to place the order for replenishment i.e., what is the optimal replenishment time?

(*iii*) How much safety stock is to be maintained i.e., what quantity of an item is to be held as buffer in anticipation of demand fluctuations or supply lags?

These are the basic questions that a decision maker has to answer to develop an optimal inventory control policy. However, before going into the further details of various problems in Inventory Control, it is essential to know the terms that are most often come across in these problems. They are therefore explained briefly in the glossary below:

Demand

It is defined as the amount of a particular good or service that a customer wants to purchase for a given price. In terms of inventory, it may be defined as the number of units of an item required by a customer and has the dimension of quantity. The demand pattern of an item may be deterministic, probabilistic, fuzzy or fuzzy random in nature.

(*i*) When the demand is known precisely and with certainty, then the demand is said to be *deterministic*. It may be constant or a variable over a period of time. Variable demand may be dependent on time, on-hand inventory, selling price etc.

(*ii*) The demand is said to be *probabilistic* when the demand is not known with certainty but its pattern can be expressed as a probability distribution.

(*iii*) The demand is called *fuzzy* when it is described in terms of non-random vague / imprecise data or in terms of linguistic variables.

(*iv*) The demand is said to be *fuzzy random* in nature when randomness and fuzziness are both present simultaneously.

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Replenishment

It is defined as the quantity that is to be put back into the inventory system at the completion of a cycle. Replenishment size may be a constant or a variable depending on time, demand, on-hand inventory level etc. Replenishment may be instantaneous (stock built up from an outside source immediately once an order is placed) or finite (quantity manufactured / procured at a finite rate).

Lead time

The lead time is the time lag between placement of an order (or initiation of a production process) and its actual arrival and addition to inventory (Hadley and Whitin 1963). It has two main components such as the administrative lead time from initiation of procurement to actual placing of the order, and the delivery lead time from the placement of the order to its actual delivery. These principal components may be further decomposed into several other components as well. The lead time may be a constant (variations in lead time small enough to be negligible) or a variable. When there is no time lag between the placement of the order and its delivery then the lead time is said to be zero.

Planning / Time Horizon

The time over which the inventory system is to be optimized i.e. the cost is to be minimized or the profit is to be maximized is called the planning / time horizon. It may be finite or infinite depending on the nature of the system.

Buffer / Safety Stock

It is the quantity of stock that is kept in reserve to act as a buffer for demand and / or price fluctuations. Mathematically it may be defined as the expected net inventory (on hand inventory for lost sales case) just prior to the arrival of an order (Hadley and Whitin 1963). The buffer stock may be positive, zero or even negative depending on the situation. But mostly the mangers / decision makers do not want their safety stock to go negative as this may cause loss of goodwill on their part.

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Constraints

They are the restrictions imposed on the inventory system in terms of the total available amount for investment, total storage space etc.

Cost Price / Purchase Cost / Production Cost

It is defined as the amount of money paid to purchase a unit of an item from an external source or the price for unit replenishment in case of internal production. It may be a constant or a variable. It may also be dependent on the quantity purchased as in the case of bulk purchasing.

Selling Price

It is the price for which a unit of an item is sold to the customer. In other words, it may be defined as the market value that will purchase a definite quantity, weight, or other measure of a good or service. It is usually a constant but in some cases may depend on the on-hand inventory level.

Ordering Cost

It is the cost associated with the issuing of orders of items to an external supplier. It includes clerical and administrative costs, telephone and postal charges, transportation costs etc. Usually it is independent of the quantity ordered. In some cases, it may depend on the quantity of goods purchased because of price break or quantity discounts or transportation cost, etc.

Setup Cost

It is the cost associated with setting up a machine, work center or assembly line before starting internal production or to switch from production to another. Setup costs include design costs, acquisition and location of machinery, paperwork cost of processing the work order, ordering cost to provide raw materials for the batch and employee hiring and training etc.

Holding / Carrying Cost

It is the cost associated with the carrying or holding of inventory. Holding cost is usually assumed to vary proportionally to the amount of inventory on hold as well as

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the time for which goods are kept in stock. It includes direct out of pocket costs like costs for handling, maintenance, insurance, rent for warehouse, electricity etc. It also includes an opportunity cost as the capital being tied up in inventory (Hadley and Whitin 1963).

Shortage / Backorder Cost

It is the cost associated with "having demands occur when the system is out of stock" (Hadley and Whitin 1963). It includes the cost for loss of goodwill, loss of profit due to lost sales, overtime / idle time payments etc. The shortage cost may be assumed to be a constant but may also be assumed to depend on the shortage quantity and the duration of time over which shortages exist.

Salvage Cost

During transportation or storage some items may become partially or fully damaged especially for perishable items. For seasonal items, the items lose their value once the selling season is over. Such items are often sold at a lower price (usually less than the purchasing cost). This is called the salvage cost.

Investment for Reducing Setup Cost

It is the investment made by the decision maker to reduce the setup cost of an inventory system. The investment may be in terms of worker training, advanced equipment acquisition, procedural changes etc. The benefits of reduced setup include reduced production lot size, yield of better quality products, increased effective capacity, shorter lead-time, increased the operational flexibility, lower unit cost etc. The investment function may be linear, exponential, piecewise linear, logarithmic etc. depending on the nature of the system.

Investment for Improving Quality

It is the investment made by the decision maker to produce better quality products. The main advantage of this investment is that it averts the occurrence of substantial costs

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arising due to the rejection, repair or refund of defective items. The investment function may be linear, piecewise linear, logarithmic etc. depending on the nature of the system.

Now the principal objective of *Operations Research* is to arrive at a '*decision*' attained by the sequential implementation of the processes of information acquisition, modeling, simulation and action. Specifically in the field of Inventory Control, this decision regarding a real life scenario needs to be executable and optimal in terms of minimizing costs or maximizing profit. But any attempt to truly model this complex real life inventory situation makes the model difficult to analyze or to conclude the process of computation in real time, thus robbing the exercise of its practicality. But in order to analyze any real life inventory situation and arrive at any implementable decision, ignoring these real life uncertainties for the sake of model building and analysis defeats the very purpose of mathematical modeling itself. So the main aim of mathematical modeling of any inventory situation is to develop a realistic representation of it while at the same time maintaining its computational feasibility and applicability. The efforts made by various researchers over the years in this regard are briefly chronicled in the following sections.

1.2 Analysis of deterministic and probabilistic inventory systems

An inventory system is concerned with the effective management of the flow of goods subject to a wide variety of constraints, e.g., demand uncertainties, seasonality, budget constraint, available storage space constraint etc., and inventory modeling is concerned with the suitable mathematical representation of these processes. Initially, inventory models were mostly developed with the consideration that all the parameters involved in the model are deterministic. In other words, it was assumed that the information regarding all the parameters defining an inventory system was known precisely with certainty. Thus the parameters were either crisp constants or crisp variables. But with the progress of time and with further development of scientific methods and tools, better and more realistic representations of inventory situations came to be studied. Researchers developed inventory models in the probabilistic environment where it was assumed that the information regarding some or all the parameters governing the inventory system is

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not known with certainty. In other words, it was assumed that the pattern followed by some or all the parameters can at best be represented by a suitable probability distribution. The parameters were therefore random variables with some probability distribution. A lot of research has been done with these two approaches for various types of inventory models. In fact, there exist thousands of research papers in the literature for the analysis of deterministic and probabilistic inventory models. A brief review of literature of inventory models developed in these two frameworks has been presented below.

The concept of analysis of an inventory system was initiated by Ford Harris (1915) of the Westinghouse Corporation, USA when he derived the classical lot size formula. This formula was also developed independently by R. H. Wilson (1934). This was a simple model formulated under assumptions such as the demand rate is uniform and known, shortages are not allowed, replenishment rate is infinite and lead-time is negligible, etc. A full length book on inventory problems was first published by Raymond (1931). This book does not contain any theory or derivations. Rather it only attempts to explain how various extensions in a simple lot size model can be used in practice. Since World War II, with the increased focus on various aspects of *Operations Research*, numerous publications have been devoted solely to the subject of inventory control. Arrow et al. (1951) presented a study on optimal inventory policy. Whitin (1957) however, was the first to put forward a theory of inventory management. Later Wagner and Whitin (1958) developed a dynamic version of the economic lot-size problem. An impressive collection on this subject up to early sixties was put together and published by Hadley and Whitin (1963) in their book '*Analysis of Inventory Systems*'. Naddor (1966) also published a book considering the mathematical properties of inventory systems. Philips et al. (1976) wrote a book entitled *Operations Research, Principle and Practice*. Since the pioneering works of these illustrious researchers, a lot of work has been done to further enhance the study and development of various types of inventory models in the deterministic and probabilistic framework. But since the work done in the thesis is mainly based on the review systems and the single period inventory model, a brief review of the literature related to the study of these models with various assumptions has been presented.

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A single period inventory problem, otherwise referred to as the 'newsboy' problem, is one of the most fundamental and widely used models in various kinds of business. The classical single period problem is to find the optimal order quantity of products which have a single replenishment opportunity before the start of the selling season so as to maximize the profit or minimize the total cost. For instance, with seasonal goods such as Christmas cards, all demand should be met by Christmas. Any card left after that has almost no value. The newsboy problem has attracted considerable attention from researchers since the initial years of the analysis of inventory systems. The classical newsboy problem and its applications can be found in Hadley and Whitin (1963) and Naddor (1966). In fact, in the real world, there exist several extensions and variations of the classical newsboy problem. For instance, Kabak and Schiff (1978) developed a single period inventory problem with the objective of maximizing the probability of achieving a target profit. Parlar and Goyal (1984) derived optimal ordering decisions for two substitutable products with stochastic demands. Lau and Lau (1988) presented a newsboy problem with price dependent demand. Baker and Urban (1988) presented a single period model with on-hand inventory dependent demand. Li et al. (1991) considered a two-product newsboy problem with independent exponential demands. Ben-Daya and Raouf 1993 and Lau and Lau (1995) considered multi-product multi-constraint newsboy problems. Gallego and Moon (1993) presented a distribution free newsboy problem. Jain and Silver (1995) extended the newsboy problem in which the supplier capacity and demand both are assumed as random variable but the newsboy can assure the availability of a given level of capacity by paying the supplier a premium ahead of time. Khouja (1996) presented a newsboy problem with an emergency supply option. Khouja et al. (1996) also studied a two-item newsboy problem with substitutability. Lau and Lau (1997) presented inventory models for single period products with reordering opportunities under stochastic demand. Urban and Baker (1997) investigated a single period model in which the demand of a product is a deterministic, multivariate function of price, time and level of inventory and there is a markdown in price. An extensive literature review on various extensions of the classical newsboy problem and related multi-stage, inventory control models was presented by Khouja (1999). Inderfurth (2004) presented a method for policy analysis in stochastic manufacturing and remanufacturing

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problems under single period framework in which there is a downward substitution policy.

Another fundamental inventory control model that has attracted the attention of researchers for a long time is the review system. For instance, Kalpakam and Arivarignan (1988) investigated a continuous review inventory system with semi-Markovian demands. Kalpakam and Sapna (1994) analyzed the model with random lifetimes and positive lead-times. Sharma (2000) developed the continuous review model for both constant and stochastic lead-time. Several other authors such as Gerchak and Gupta (1991), Singal et al. (1994), Zhang et al. (2003), Ghalebsaz-jaddi et al. (2004), Pan et al. (2004) have also studied this model under various considerations. Ertogral and Rahim (2005) presented a replenish-up-to inventory level policy with random replenishment intervals. Isotupa (2006) analyzed a continuous review inventory system with exponential lead-time and ordinary and priority based customers. In the case of periodic review systems, several researchers like Chiang (2001, 2007), Chan and Song (2003), Li et al. (2004), Gavirneni (2004), Bylka (2005), Konstantaras and Papachristos (2007), Eynan and Kropp (2007) have reviewed the periodic review system under different conditions in the probabilistic framework. For the continuous review model, the model has been studied by researchers for the backorders case, the lost sales as well as for a mixture of the two. The continuous review model with a mixture of backorders and lost sales was first studied by Montgomery et al. (1973). Quyang and Wu (1996) analyzed a similar model for variable lead time with fixed reorder point. Many researchers like Kim and Park (1985), Hariga and Ben Daya (1999), Abad (2000), Kumaran et al. (2006) have also studied such mixture models with various assumptions. This model has also been analyzed with the aim of lead-time reduction by researchers like Liao and Shyu (1991), Ben-Daya and Raouf (1994), Ouyang et al. (1996) etc. The problem of lead-time reduction for the periodic review system has also been studied recently by some researchers like Ouyang and Chuang (2001), Chuang et al. (2004) with various assumptions. The analysis of the continuous review model with setup cost reduction, introduced by Porteus (1985) and product quality improvement, introduced by Porteus (1986) and Rosenblatt and Lee (1986) etc. have also attracted considerable attention from

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researchers over the years. For instance researchers like Moon and Choi (1998), Hariga and Ben-Daya (1999), Pan and Hsiao (2005), Chang (2005), Chang and Lo (2009) have studied this problem of lead-time reduction. Keller and Noori (1988), Hwang et al. (1993), Hong et al. (1993), Moon (1994), Hong and Hayya (1995) and Voros (1999) etc. have analyzed the problem of quality improvement and researchers like Lee et al. (1997), Kim and Hong (1999), Hou and Lin (2004), Hou (2007), Kulkarni (2008) etc. have taken up the issue of setup cost reduction in various kinds of inventory models. Recently, Ouyang et al. (2002) has analyzed the combined effects of lead-time reduction, setup cost reduction and quality improvement in the lot size reorder point model.

As evident from the brief review of literature mentioned above, inventory modeling in the deterministic and probabilistic framework has received a lot of attention from researchers over a considerable length of time. With further development of mathematical analysis, another kind of inventory modeling dealing with a non-probabilistic uncertainty has also come to be studied. This uncertainty does not arise due to any random fluctuation but rather due to partial information about the inventory situation (for instance, lack of demand information for new products), inherent vagueness in the language in which the situation is defined (for instance, an expert predicts the demand of a product to be 'about *D'*), or due to the receipt of information from more than one source about the problem which is conflicting. Inventory models based on this kind of non-probabilistic uncertainty – called imprecision or fuzziness – are called fuzzy inventory models. In the next section, the definitions, mathematical operations and the tools required for developing these fuzzy inventory models have been discussed and a brief review of literature presented.

1.3 Fuzzy set theory and related inventory models

1.3.1 Preliminary concepts of fuzzy set theory

The theory of fuzzy sets was first proposed by L. A. Zadeh (1965) in his now pioneering paper 'Fuzzy Sets' (Zadeh, L. A. (1965) 'Fuzzy Sets', *Information and Control,* 8, 338-353) to represent sources of imprecision or vagueness which are non-statistical in nature. Since then a lot of attention has been focused on the development of this intuitive and somewhat subtle theory as it provides an excellent mathematical tool to handle

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uncertainty arising due to imprecision. An in depth study of fuzzy sets and its applications may be found in the books by Dubois and Prade (1980), Zimmermann (1991), Ross (1997) etc. The concepts and mathematical operations used to model decision making situations with the theory of fuzzy set have been discussed below:

Definition 1.3.1 Crisp set

A set is a well defined collection of distinct elements or objects. In terms of characteristic function, a crisp set may be defined as follows:

Let X be the universal set and A be any subset of X. Then the characteristic function of A is denoted by χ_A and is defined by the mapping $\chi_A : X \to \{0,1\}$ where

$$\chi_A(x) = \begin{cases} 1, & iff \quad x \in A \\ 0, & iff \quad x \notin A \end{cases}$$

For example, let us consider the crisp set A where A stands for the set of all real numbers lying between 1 and 10. In other words,

$$A = \{x : 1 \le x \le 10, x \in \mathbb{R}\}$$

Then the characteristic function of this crisp set A is given as a mapping $\chi_A : \mathbb{R} \to \{0,1\}$ where

$$\chi_A = \begin{cases} 1, & x \in [1,10] \\ 0, & \text{otherwise} \end{cases}$$

From the above definition it is obvious that for a crisp set, either an element belongs to a set (for which the characteristic function is 1) or does not belong to the set (for which the characteristic function is 0).

Definition 1.3.2 Fuzzy set

For the case of a fuzzy set, there is no clear sense of belongingness or non-belongingness. This statement becomes clear from the following definition of a fuzzy set. Let X be the universe of discourse under consideration and $\tilde{A}$ be a fuzzy set. Then this fuzzy set $\tilde{A}$ is

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defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. Here $\mu_{\tilde{A}}(x)$ is a mapping $\mu_{\tilde{A}}(x) : X \rightarrow [0,1]$ and is called the membership function of the fuzzy set $\tilde{A}$. For instance, $\tilde{A}$ may be the set of 'beautiful girls' in a particular class. Since 'beauty' is a subjective concept, such a collection can not be truly represented by a crisp set. This is where fuzzy set theory comes into play capturing subjective or linguistic evaluation and vagueness of the situation.

It is to be noted here that while the characteristic function for a crisp set could be either 0 (for non-belonging) or 1 (for belonging), the membership function takes values on the closed interval $[0,1]$. Therefore there is no sharp boundary between those objects that belong to the set and those that do not. $\mu_{\tilde{A}}(x)$ represents the degree of membership or belongingness to the set. So larger the value of $\mu_{\tilde{A}}(x)$, greater is the degree of belongingness and vice versa. The membership function $\mu_{\tilde{A}}(x)$ may be either discrete or continuous. When the universe of discourse is a finite set $X = \{x_1, x_2, ..., x_n\}$, then a fuzzy set $\tilde{A}$ defined on X may be represented as

$$A = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + ... + \mu_{\tilde{A}}(x_n)/x_n = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i$$

For an infinite universe of discourse X, $\tilde{A}$ may be represented as $A = \int_X \mu_{\tilde{A}}(x)/x$.

Definition 1.3.3 Support of a fuzzy set

The support of a fuzzy set $\tilde{A}$ is a crisp subset of the universe of discourse X defined as

$$\text{supp } \tilde{A} = \{x \in X : \mu_{\tilde{A}}(x) > 0\}$$

Definition 1.3.4 α-level set of a fuzzy set

The α-level set of a fuzzy set $\tilde{A}$ is a crisp subset of the universe of discourse X denoted by

$$A(\alpha) = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\} \quad \forall \alpha \in [0,1]$$

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Definition 1.3.5 Fuzzy set theoretic operations

Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ be fuzzy sets defined on the universe of discourse X. For any given element $x \in X$, the following operations are defined as follows:

Union: $\tilde{C} = \tilde{A} \cup \tilde{B}$ iff $\mu_{\tilde{C}}(x) = \max\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right), \quad \forall x \in X$

Intersection: $\tilde{D} = \tilde{A} \cap \tilde{B}$ iff $\mu_{\tilde{D}}(x) = \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right), \quad \forall x \in X$

Containment: $\tilde{A} \subset \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x) \quad \forall x \in X$

Equality: $\tilde{A} = \tilde{B}$ iff $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \quad \forall x \in X$

Complement: $\tilde{E} = \tilde{A}^C$ iff $\mu_{\tilde{E}}(x) = 1 - \mu_{\tilde{A}}(x), \quad \forall x \in X$

Definition 1.3.6 Convex fuzzy set

A fuzzy set $\tilde{A}$ is said to be convex if and only if its α-level sets are convex for all $\alpha \in [0,1]$. Equivalently a fuzzy set $\tilde{A}$ may be defined to be convex if and only if

$$\mu_{\tilde{A}}\left(\zeta x_1 + (1-\zeta)x_2\right) \geq \min\left(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right), \quad \forall x_1, x_2 \in X, \ \forall \ \zeta \in [0,1]$$

Definition 1.3.7 Normal fuzzy set

A fuzzy set $\tilde{A}$ defined on the universe of discourse X is called a normal fuzzy set if and only if there exist at least one $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

Definition 1.3.8 Fuzzy number

A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set defined on the universe of discourse $\mathbb{R}$ (the set of all real numbers) with a piecewise continuous membership function and bounded support.

Definition 1.3.9 L-R Representation of fuzzy number

A fuzzy number $\tilde{A}$ is said to be an L-R type fuzzy number (Dubois and Prade 1980) iff

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$$\mu_{\tilde{A}}(x) = \begin{cases} \mathrm{L}((m-x)/\Delta_1), & \text{for} \quad x \le m \quad \text{and} \quad \Delta_1 > 0 \\ \mathrm{R}((x-m)/\Delta_2), & \text{for} \quad x \ge m \quad \text{and} \quad \Delta_2 > 0 \end{cases}$$

L and R are called the left and right reference functions, m is the mean and Δ_1, Δ_2 the left and right spreads, respectively.

These reference functions satisfy the following conditions:

(i) $\mathrm{L}(x) = \mathrm{L}(-x)$

(ii) $\mathrm{L}(0) = 1$

(iii) $\mathrm{L}(\cdot)$ is non-increasing on $[0,\infty)$

The left and right shape functions may be both linear and non-linear. For example, $\mathrm{L}(x) = 1, x \in [-1,1]$, $\mathrm{L}(x) = e^{-|x|^p}, p \ge 0$, $\mathrm{L}(x) = \max(0, 1-|x|^p), p \ge 0$ are instances of the left reference function. The right reference function may be defined similarly. The fuzzy number $\tilde{A}$ is denoted as $\tilde{A} = (m, \Delta_1, \Delta_2)_{\mathrm{LR}}$.

Definition 1.3.10 **Generalized fuzzy number**

A generalized fuzzy number $\tilde{A} = (a,b,c,d;w)$ with grade w (Chen and Chen 2003) is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0,1]$

(ii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$

(iii) $\mu_{\tilde{A}}(x) = w$, $x \in [b,c]$

(iv) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $[c,d]$

(v) $\mu_{\tilde{A}}(x) = 0$, elsewhere

where $0 < w \le 1$ and a,b,c,d are real numbers. When w=1, it is called a generalized normal fuzzy number. Thus, a generalized normal fuzzy number is denoted as $\tilde{A} = (a,b,c,d)$.

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Definition 1.3.11 Triangular fuzzy number

A normalized triangular fuzzy number $\tilde{A} = (\underline{a}, a, \overline{a})$, where $\underline{a}, a, \overline{a}$ are real numbers, is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0,1]$

(ii) $\mu_{\tilde{A}}(x) = \dfrac{x - \underline{a}}{a - \underline{a}}, \quad \underline{a} \le x \le a$ is strictly increasing on $[\underline{a}, a]$

(iii) $\mu_{\tilde{A}}(x) = 1, \; x = a$

(iv) $\mu_{\tilde{A}}(x) = \dfrac{\overline{a} - x}{\overline{a} - a}, \quad a \le x \le \overline{a}$ is strictly decreasing on $[a, \overline{a}]$

(v) $\mu_{\tilde{A}}(x) = 0$, elsewhere

where $\underline{a}, a, \overline{a}$ are real numbers. A normal triangular fuzzy number $\tilde{A}$ may also be denoted by its α-cut as $A = [A_\alpha^-, A_\alpha^+]$, where $\alpha \in [0,1]$. The α-cut representation and the triangular notation are then connected as follows:

$$A_\alpha^- = \underline{a} + \alpha(a - \underline{a}) \text{ and } A_\alpha^+ = \overline{a} - \alpha(\overline{a} - a)$$

The triangular fuzzy number may also be expressed as an L-R type fuzzy number as

$$\tilde{A} = (m, \Delta_1, \Delta_2)_{\text{LR}} \text{ where } m = a, \Delta_1 = a - \underline{a}, \Delta_2 = \overline{a} - a.$$

Without any loss of generality, all fuzzy quantities have been assumed to be normal triangular fuzzy numbers of the form $\tilde{A} = (\underline{a}, a, \overline{a})$ with $\underline{a}, a, \overline{a}$ as positive real numbers throughout this thesis.

Definition 1.3.12 Defuzzification of a fuzzy number

For the sake of computational simplicity, researchers have introduced various means of determining a crisp representation of a fuzzy number. Two of such methods that have been employed in the thesis are the Graded mean integration representation of a fuzzy number proposed by Chen and Hsieh (1999) and the Possibilistic mean of a fuzzy number presented by Carlsson and Fuller (2001). These two methods have been discussed briefly below:

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- ***Graded mean integration representation of a fuzzy number***

The Graded Mean Integration Representation (GMIR) of a fuzzy number was proposed by Chen and Hsieh (1999) as a means of defuzzifying a generalized fuzzy number $\tilde{A}=(a,b,c,d;w)$ by incorporating the grade w of each point of the support set of the fuzzy number. However since a normalized ($w=1$) triangular number has been used throughout the thesis, therefore the derivation has been illustrated with $w=1$.

Let L^{-1} and R^{-1} be the inverse functions of the left and right shape functions L and R, respectively, of a fuzzy number $\tilde{A}=\left(\underline{a},a,\overline{a}\right)$ say. Then the graded mean α-level value of the fuzzy number $\tilde{A}=\left(\underline{a},a,\overline{a}\right)$ is given by $\alpha[\mathrm{L}^{-1}(\alpha)+\mathrm{R}^{-1}(\alpha)]/2$. Therefore, the graded mean integration representation of a normalized triangular fuzzy number $\tilde{A}$ is given by

$$G\left(\tilde{A}\right)=\int_0^1\left(\alpha\left[\mathrm{L}^{-1}(\alpha)+\mathrm{R}^{-1}(\alpha)\right]/2\right)d\alpha\Bigg/\int_0^1\alpha\,d\alpha$$

where α lies between 0 and 1.

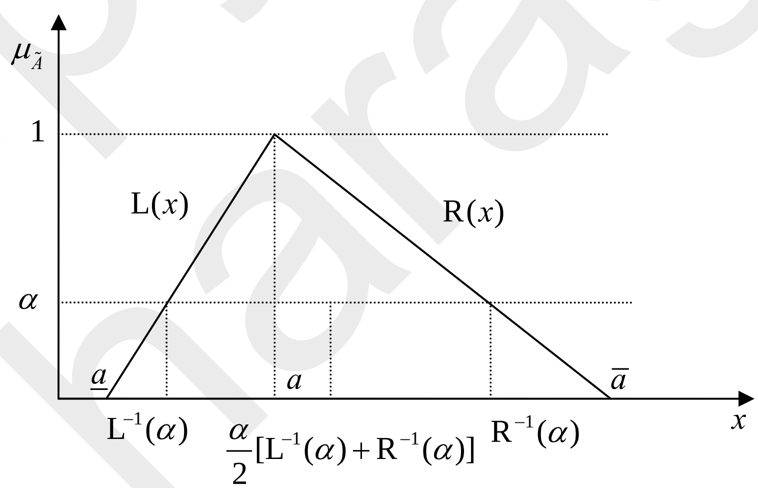

Fig 1.1 Graded α-level value of a triangular fuzzy number $\tilde{A}=\left(\underline{a},a,\overline{a}\right)$

Now, from Definition 1.3.10, the left and right shape functions for a triangular fuzzy number are given respectively by

$$\mathrm{L}\left(x\right)=\left(\frac{x-\underline{a}}{a-\underline{a}}\right),\quad \underline{a}\leq x\leq a$$

$$\mathrm{R}\left(x\right)=\left(\frac{\overline{a}-x}{\overline{a}-a}\right),\quad a\leq x\leq \overline{a}$$

Thus,

$$\mathrm{L}^{-1}\left(\alpha\right)=\underline{a}+\left(a-\underline{a}\right)\alpha$$

$$\mathrm{R}^{-1}\left(\alpha\right)=\overline{a}-\left(\overline{a}-a\right)\alpha$$

Therefore using the formula proposed by Chen and Hseih (1999) the graded mean integration representation of $\tilde{\mathrm{A}}$ is given by

$$G\left(\tilde{\mathrm{A}}\right)=\int_0^1(\alpha[\underline{a}+\left(a-\underline{a}\right)\alpha+\overline{a}-\left(\overline{a}-a\right)\alpha]/2)d\alpha\,/\int_0^1\alpha d\alpha=\frac{\underline{a}+\overline{a}}{6}+\frac{2a}{3}$$

Another method that provides a means of deriving a crisp equivalent of a given fuzzy number is the crisp possibilistic mean value and crisp possibilistic variance of continuous possibility distributions (membership functions) proposed by Carlsson and Fuller (2001).

- ***Possibilistic mean value of a fuzzy number***

For a given fuzzy number $\tilde{A}$, the interval-valued possibilistic mean was formulated as a theory of continuous possibility distributions "which are consistent with the extension principle and with the well-known definitions of expectation and variance in probability theory" (Carlsson and Fuller 2002). In this regard, the interval-valued possibilistic mean is defined as

$$M(\tilde{A})=[M_*(\tilde{A}),M^*(\tilde{A})]$$

where $M_*(\tilde{A})$ and $M^*(\tilde{A})$ are the lower and upper possibilistic mean values of $\tilde{A}$. The first quantity $M_*(\tilde{A})$ is the lower possibility-weighted average of the minima of the $\alpha-$sets of $\tilde{A}$ and is defined as

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$$M_*(\tilde{A}) = \frac{\int_0^1 \mathrm{Pos}[\tilde{A} \le A_\alpha^-] A_\alpha^- d\alpha}{\int_0^1 \mathrm{Pos}[\tilde{A} \le A_\alpha^-]\, d\alpha} = \frac{\int_0^1 \alpha A_\alpha^- d\alpha}{\int_0^1 \alpha d\alpha}$$

where Pos denotes possibility, i.e.,

$$\mathrm{Pos}[\tilde{A} \le A_\alpha^-] = \sup_{x \le A_\alpha^-} A(x) = \alpha$$

Similarly, the upper possibility-weighted average of the maxima of the $\alpha-$sets of $\tilde{A}$ is defined as

$$M^*(\tilde{A}) = \frac{\int_0^1 \mathrm{Pos}[\tilde{A} \ge A_\alpha^+] A_\alpha^+\, d\alpha}{\int_0^1 \mathrm{Pos}[\tilde{A} \ge A_\alpha^+]\, d\alpha} = \frac{\int_0^1 \alpha A_\alpha^+ d\alpha}{\int_0^1 \alpha d\alpha},$$

where Pos denotes possibility, i.e.,

$$\mathrm{Pos}[\tilde{A} \ge A_\alpha^+] = \sup_{x \ge A_\alpha^+} A(x) = \alpha$$

Therefore, the possibilistic mean value of $\tilde{A}$ is defined as the arithmetic mean of its lower and upper possibilistic mean values, i.e.,

$$\bar{\mathrm{M}}(\tilde{A}) = \frac{M_*(\tilde{A}) + M^*(\tilde{A})}{2}$$

In other words, one can write the level-weighted average of the arithmetic mean of all $\alpha-$level sets of $\tilde{A}$ as

$$\bar{M}(\tilde{A}) = \frac{\int_0^1 (\alpha(A_\alpha^- + A_\alpha^+)/2\,)d\alpha}{\int_0^1 \alpha\, d\alpha} = \int_0^1 \alpha(A_\alpha^- + A_\alpha^+)\, d\alpha$$

For a normal triangular fuzzy number $\tilde{A} = (\underline{a}, a, \bar{a})$, the possibilistic mean is given as:

$$\bar{\mathrm{M}}(\tilde{A}) = \int_0^1 \alpha(A_\alpha^- + A_\alpha^+) d\alpha = \int_0^1 \alpha(\underline{a} + \alpha(a - \underline{a}) + \bar{a} - \alpha(\bar{a} - a)) d\alpha = \frac{\underline{a} + \bar{a}}{6} + \frac{2a}{3}$$

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It is to be noted here that the two methods outlined above are essentially different in their intended purpose for which they were developed. While the GMIR of a fuzzy number (Chen and Hseih 1999) was developed essentially as a means of defuzzifying generalized fuzzy numbers, the possibilistic mean was proposed to develop a theory for continuous possibility distribution which was analogous to the existing theory of continuous probability distribution. However as regards the defuzzification of a normalized triangular fuzzy number, both the GMIR method and the possibilistic mean yield the same result. So they can be used interchangeably for defuzzification purposes.

Based on the concept of fuzzy set theory, as briefly discussed above, researchers developed various inventory models incorporating the imprecision of the inventory parameters into the model building exercise. A brief review of literature in this regard has been presented below.

1.3.2 Literature review of fuzzy inventory models

In any real life decision making situation, proper encoding of the information regarding the various parameters governing the situation is of utmost importance. Often, due to lack of precise data, aggregation of data and variations over time, presence of linguistic expressions etc. (Oder et al. 1993), some vagueness in encoding the parameters may arise. In order to tackle such non-statistical uncertainty, Zadeh (1965) first introduced fuzzy set theory proposing new techniques to accommodate such uncertainty in the non-stochastic sense. Various researchers like Bellman and Zadeh (1970), Dubois and Prade (1978), Zimmermann (1985), Kaufmann and Gupta (1985) etc. have also developed fuzzy set theoretic approaches and investigated their use in developing mathematical representation of real life decision making situations. Since then, the fuzzy approach has been employed widely for modeling various real life decision making situations including inventory control.

Sommer (1981) used fuzzy dynamic programming to solve an inventory and production-scheduling problem in which the management wishes to fulfill a contract for providing a product and then withdraws from the market. Park (1987) and Vujosevic et al. (1996)

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developed the fuzzy economic order quantity (EOQ) model by introducing fuzziness of ordering cost and holding cost. Roy and Maiti (1995) studied a fuzzy inventory model with constraints. Lam and Wong (1996) solved the joint economic lot-size problem with multiple price breaks using fuzzy mathematical programming. Chen and Wang (1996) fuzzified the demand, ordering cost and backorder cost in a backorder fuzzy inventory model under function principle. Roy and Maiti (1997) solved fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. Chang et al. (1998) presented a fuzzy inventory model with backorder with the backorder quantity was fuzzified as the triangular fuzzy number. Lee and Yao (1998) presented an economic production quantity (EPQ) model with fuzzy demand quantity and fuzzy production quantity. Lin and Yao (2000) developed a fuzzy economic production for production inventory. De and Goswami (2001) presented an EPQ model with fuzzy deterioration rate. Mandal and Maiti (2002) solved a fuzzy EOQ model using genetic algorithm. Hsieh (2002) developed a fuzzy production inventory model where all parameters are treated as fuzzy numbers along with the annual demand assumed to be fuzzy as well. Yao and Chiang (2003) developed an inventory model without backorder with fuzzy total cost and fuzzy storing cost. Wu and Yao (2003) investigated a backorder inventory model assuming fuzzy order quantity and fuzzy shortage quantity. Yao et al. (2003) developed a fuzzy inventory model of two replaceable merchandises without backorder. Mahata et al. (2005) presented a joint economic lot-size model for purchaser and vendor with fuzzy order quantity. Mandal et al. (2005) solved a multi-objective inventory problem using the geometric programming approach. Mahata et al. (2006) presented a production lot-size model with fuzzy production rate and fuzzy demand rate for deteriorating items under permissible delay in payments. Maiti and Maiti (2007) developed a two-storage fuzzy inventory model under possibility constraints and lot-size dependent fuzzy lead-time. A fuzzy economic order quantity model with fuzzy profit measures was studied by Liu (2008). Rong et al. (2008b) investigated a two warehouse inventory model for a deteriorating item with partially/fully backlogged shortage and fuzzy lead time. Maiti (2008) presented a fuzzy inventory model with two warehouses under possibility measure on fuzzy goal. Chen and Chang (2008) analyzed the optimization of fuzzy production inventory model with unrepairable defective products. In this model, they considered a

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fuzzy opportunity cost and trapezoidal fuzzy costs under crisp production quantity or fuzzy production quantity in order to extend the traditional production inventory model to the fuzzy environment. Wee et al. (2009) investigated a multi-objective joint replenishment inventory model of deteriorated items in a fuzzy environment. Vijayan and Kumaran (2009) studied the fuzzy economic order time models with random demand with the time period of sales considered to be a decision variable assuming the components of the model as fuzzy sets. The arrival of customers and the number of customers in the planning period are both assumed to be random. Bjork (2009) presented an analytical solution to fuzzy economic quantity problem. Chou et al. (2009) presented note on fuzzy inventory model with storage space and budget constraints. In this paper they pointed out some questionable results presented by Roy and Maiti (1997) and derived improved solution procedures. Roy et al. (2009a) also presented a production inventory model with remanufacturing for defective and usable items in fuzzy environment. Roy et al. (2009b) analyzed an inventory model for a deteriorating item with displayed stock dependent demand under fuzzy inflation and time discounting over a random planning horizon. As evident from the literature review presented above, a lot of research has been carried out by various researchers for the development of fuzzy inventory models.

A newsboy type problem with fuzzy demand was first investigated by Petrovic et al. (1996). Ishii and Konno (1998) presented a single period inventory model with random demand and fuzzy shortage cost. Li et al. (2002) and Kao and Hsu (2002a) independently developed the single period inventory model with fuzzy demand. Later Ji and Shao (2006a, 2006b) developed a model and presented an algorithm for a bilevel newsboy problem with fuzzy demands and also investigated the fuzzy multi-product constraint newsboy problem. Dutta et al. (2007a) analyzed an inventory model for single period products with reordering opportunities assuming the demand to be fuzzy. Panda et al. (2008) developed a mathematical model for a single period multi-product manufacturing system of stochastically imperfect items with continuous stochastic demand and fuzzy possibility and necessity constraints.

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The review systems have also received a lot of attention from researchers investigating the applicability of fuzzy set theory in modeling various inventory situations. Chang (2001) studied the lead-time reduction model based on the periodic review system with fuzzy backorders. Kao and Hsu (2002b) investigated a lot-size reorder point model with fuzzy demand. A minimax distribution free procedure for mixed inventory models involving variable lead time was developed by Ouyang and Chang (2002) assuming fuzzy lost sales. A minimax distribution free for mixed inventory model involving random variable lead time demand was then analyzed by Ouyang and Yao (2002) with fuzzy demand. Kumaran and Vijayan (2008) developed both the continuous and the periodic review systems under fuzzy costs. Especially with the increasing emphasis on the development of supply chain management, the application of fuzzy set theory in the supply chain has garnered attention by researchers like Petrovic and Sweeney (1994), Xie et al. (2006), and Petrovic et al. (2008) etc.

Thus, as evident from the literature above, fuzzy inventory models have been focused on by various researchers in the recent past.

Now although, as evident from sections 1.2 and 1.3.2, a vast amount of literature is available on purely probabilistic and purely fuzzy inventory models, amalgamation of these two existing approaches is an issue that remains to be looked into. In fact, this simultaneous existence of imprecision and uncertainty in a single parameter in any real life decision making situation (Chakraborty 1995) is a common phenomenon and the mathematical realization of the fact has already been a field of interest among researchers. For instance, pertaining to the issue of inventory management under consideration, the demand information may be collected from various experts who express their opinions n terms of evaluative linguistic expressions like ‘about’, more or less’, ‘roughly’ etc. That is, one expert says the ‘the demand is around D_1’, another that ‘the demand is more or less D_2’ and so on. Now these are linguistic expressions which vary randomly from one expert to another and so the simultaneous presence of fuzziness and randomness is obvious. However the problem of integrating these two concepts – uncertainty and imprecision – in an inventory control setting is not yet attended to so well and this is the aspect that has motivated the present work. The basic differences between these two types of uncertainty and the reasons as to why they should be considered simultaneously have now been discussed in the subsequent section.

1.4 The fuzzy random environment and related inventory models

1.4.1 Fundamentals of fuzzy randomness

The two kinds of uncertainties – fuzziness and randomness – are practically and theoretically different though they both use the unit interval $[0,1]$ as their measure. Imprecision arises due to partial information about the problem, aggregation of data to reduce size of the model, inherent vagueness in the language in which the problem is defined or the receipt of information from more than one source about the problem which is conflicting etc. On the other hand randomness occurs due to uncertainty regarding the occurrence of an event and not due to any vagueness of the event description i.e., in probability theory, everything is by definition well defined and the notion of probability is required only to quantify our ignorance (uncertainty) about which particular case is going to be observed. In other words, fuzziness means imprecision of meaning of an object while randomness means uncertainty regarding the occurrence of an object. Therefore there is no conflict between these two types of uncertainties and the use of one does not preclude the other. With this point of view, researchers have developed a number of tools of simultaneous quantification of fuzziness and randomness. One of the most widely used of these tools is what is known as a *fuzzy random variable*.

A fuzzy random variable may be thought of as random variables whose values are fuzzy (e.g., demand is a denoted by a random variable taking on values on the set of all fuzzy numbers) or as fuzzy variables with random parameters (e.g., demand is a fuzzy number whose membership function contains random parameters). In this context, an overview of fuzzy random variable and a few of its related concepts have been presented below:

Definition 1.4.1 **Fuzzy random variable**

Kwakernaak (1978) was the first to introduce fuzzy random variables in his now classic work ‘Fuzzy Random Variables – I. Definitions and Theorems’ (Information Sciences,

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15, 1- 29). He further extended this work in 1979 in the paper 'Fuzzy Random Variables – II. Algorithms and examples for the discrete case' (Information Sciences, 17, 253-278). Puri and Ralescu (1986), Kruse and Meyer (1987) also discussed this concept in later years. Since then various researchers have focused their attention on further development of this concept. Recently Gil et al. (2006) have provided a detailed study of the various definitions of fuzzy random variable proposed for modeling various situations by researchers. In this thesis the concept of a fuzzy random variable as proposed by Kwakernaak (1978, 1979) and further formalized by Kruse and Meyer (1987) has been used. This concept, mentioned in Gil et al. (2006), has been briefly discussed below:

Let us consider the p-dimensional Euclidean space $\mathbb{R}^p$. $F\left(\mathbb{R}^p\right)$ denotes the class of upper semi continuous function in $\left[0,1\right]^{\mathbb{R}^p}$ with compact closure of the support. Then, for the one-dimensional case, $F_C\left(\mathbb{R}\right)$ is the sub-class of convex sets of $F\left(\mathbb{R}\right)$. Given a probability space $\left(\Omega, B, \mathrm{P}\right)$, a mapping $\chi : \Omega \to F_C\left(\mathbb{R}\right)$ is said to be a fuzzy random variable if for all $\alpha \in \left[0,1\right]$, the two real-valued mappings

$$\inf \chi_\alpha : \Omega \to \mathbb{R} \quad \text{and} \quad \sup \chi_\alpha : \Omega \to \mathbb{R}$$

(defined so that for all $\omega \in \Omega$ we have $\chi_\alpha\left(\omega\right) = \left[\inf\left(\chi\left(\omega\right)\right)_\alpha, \sup\left(\chi\left(\omega\right)\right)_\alpha\right]$) are real-valued random variables.

A fuzzy random variable may also be defined as $\tilde{D}(\omega) = (D(\omega) - \Delta_1, D(\omega), D(\omega) + \Delta_2)$ (Liu and Liu 2003) where $\omega \in \Omega$ and (Ω, B, P) is a probability space. Here Δ_1, Δ_2 are the left and right spreads respectively with $0 < \Delta_1 < D(\omega)$ and $\Delta_2 > 0$ for all $\omega \in \Omega$, where $D(\omega)$ follows some continuous distribution.

Definition 1.4.2 Expectation of a fuzzy random variable

The fuzzy expectation of a fuzzy random variable is a unique fuzzy number. It is defined as

$$\mathrm{E}\tilde{X}(\omega) = \int_{\Omega} \tilde{X}(\omega)dP = \left\{ \left[\int_{\Omega} X_{\alpha}^{-}(\omega)dP, \int_{\Omega} X_{\alpha}^{+}(\omega)dP \right] / \, 0 \le \alpha \le 1 \right\},$$

where the fuzzy random variable is $\left[X\right]_{\alpha} = \left[X_{\alpha}^{-}, X_{\alpha}^{+}\right], \alpha \in [0,1]$.

The α -cut of the fuzzy expectation is given by

$$\left[\mathrm{E}\tilde{X}(\omega)\right]_{\alpha} = \mathrm{E}\left[X(\omega)\right]_{\alpha} = \left\{ \left[\mathrm{E}\left(X_{\alpha}^{-}(\omega)\right), \mathrm{E}\left(X_{\alpha}^{+}(\omega)\right) \right], \alpha \in [0,1] \right\}.$$

If $\tilde{X}$ is a fuzzy random variable such that $P(\tilde{X} = \tilde{x}_i) = \tilde{p}_i$ for $i = 1, 2, 3 .... n$ then its fuzzy expectation is given as follows:

$$\mathrm{E}\tilde{X} = \sum_{i=1}^{n} \tilde{x}_i \tilde{p}_i$$

It has also been proved by Gil and Lopez-Diaz (1998) that

$$\mathrm{E}\tilde{X}_{\alpha=0} = \int_{\Omega} X_{\alpha=0} \; \mathrm{d}P = [\mathrm{E}X_{\alpha=0}^{-}, \mathrm{E}X_{\alpha=0}^{+}].$$

Now the representation of inventory parameters based on the concept of fuzzy random variables presented above has been a recent interest among researchers. A review of literature in this regard has been presented in the next subsection.

1.4.2 Literature review of fuzzy random inventory models

Of late a few researchers have made some inputs regarding the possible ways of applying the amalgamation of fuzziness and randomness in inventory modeling. For instance, the encoding of information regarding the inventory parameters, where fuzziness and randomness co-exist, by means of a fuzzy random variable was illustrated by Dutta et al. (2005) who presented a single period inventory model with a fuzzy random variable demand of the form $(\tilde{D}_i, \tilde{p}_i), i = 1, 2, ..., n$. For the problem of lead-time reduction, in the continuous review system, Chang et al. (2006) assumed the lead-time demand to be a fuzzy interval $[\mu L - \Delta_1, \mu L + \Delta_2]$ where μ is the mean of the annual demand. Yao et al. (2006) developed a fuzzy random single period model for cash management with the demand as $\tilde{D}$ and an associated fuzzy probability density function. Dutta et al. (2007b)

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presented a continuous review inventory model with annual demand assumed to be a fuzzy random variable $(\tilde{D}_i, p_i), i=1,2,...,n$. Lin (2008) assumed the expected shortage to be a fuzzy random variable of the triangular form $(x-\Delta_1, x, x+\Delta_2)$ in the periodic review system while assuming the backorder rate to be fuzzy. Rong et al. (2008a) presented a multi-objective wholesaler-retailer inventory-distribution model with controllable lead-time based on probabilistic fuzzy set and triangular fuzzy number. Xu and Liu (2008) developed a fuzzy random multi-objective model about inventory problems. Hasuike and Ishii (2009) presented flexible product-mix problems under randomness and fuzziness. Taleizadeh et al. (2009) analyzed a hybrid method of Pareto, TOPSIS and genetic algorithm to optimize multi-product multi-constraint inventory control systems with random fuzzy replenishments. The annual demand has also been assumed to be a fuzzy random variable of the form $(\tilde{D}_i, p_i), i=1,2,...,n$ by Bag et al. (2009).

Thus, as seen from the literature review presented above, though some advances have been made regarding the development of inventory models under the fuzzy random framework, much however, remains to be studied.

1.5 Motivation of the work

The main motivation for the thesis is the amalgamation of the two different types of uncertainty – fuzziness and randomness – with special emphasis to inventory management problems. The principal focus of the thesis is therefore to develop inventory models under this mixed fuzzy random framework.

One of the most widely used methods of achieving the aforementioned combination of the two approaches is by means of a fuzzy random variable. But, as evident from the review of literature presented in the previous section, though a few researchers of late have been employing this method to improve on the realistic representation of the inventory models, a lot of scope still remains. In other words, there are quite a large number of inventory models that are yet to be analyzed using fuzzy random variable to model the various inventory parameters. For instance, one of the most important

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inventory parameter, namely, the annual customer demand has mostly been assumed to be either probabilistic or fuzzy in nature. In fact, though some researchers have employed fuzzy random variable to model inventory parameters like lead-time demand, shortage etc., only a few may be referred to who have analyzed inventory models assuming the annual customer demand to be a fuzzy random variable. But it is quite well known that for the analysis of any inventory situation, one of the major difficulties faced by the decision maker is to properly code the customer demand information. According to Naddor (1966, p 21), “Demand property is the most important component in the analysis of an inventory system. Inventories are kept so that demands may be met, orders filled, requirements satisfied. Inventory problems exist only because there are demands; otherwise, we have no inventory problems”. So the problem of suitable quantification of the customer demand for an inventory system is of utmost importance.

Now the information regarding the customer demand is collected from various experts or sources. This information therefore invariably contains some amount of imprecision and uncertainty simultaneously. For instance for the case of new products, where there is a distinct lack of historical information in terms of statistical data, the customer demand is approximately estimated and / or expressed linguistically by experts. In such situations, instead of modeling the demand by a random variable (which excludes the fuzziness) or a fuzzy variable (which does not consider the inherent statistical nature of the data), the customer demand may be effectively quantified as a discrete fuzzy random variable. This assumption takes into account both the types of uncertainties. Besides, the amount of information available being less, a discrete form suffices quite efficiently. Even for inventory situations where there maybe sufficient or even abundant statistical data, this statistical data may itself contain fuzziness in terms of lack of precise information (linguistic information), inaccurate or unreliable information (lack of proper documentation), non-constant reproduction conditions (market fluctuations, seasonal variability) etc. So, keeping in mind its computational advantages, such information may be quantified using continuous fuzzy random variable instead of a fuzzy variable, a random variable or even a discrete fuzzy random variable.

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Also special attention needs to be paid if these models, which have been developed for particular markets are applied elsewhere. For instance an inventory model developed for, say, winter garments for the New Delhi market, will not be applicable in Chennai. This is not a matter of probability but the differences in their climatic conditions. Therefore, a methodology is needed which is able to deal with subjective evaluations of the decision maker. In this regard, the fuzzy random environment not only provides a much more generalized framework of representing reality, but also helps in developing a methodology that is capable of dealing with such subjectivities, in allowing the decision maker to include his knowledge and experience into the model building process and thereby, in providing an effective decision support.

With this point of view, in the thesis, inventory models have been analyzed with the annual customer demand assumed to be a fuzzy random variable. Further, as any imprecision and uncertainty in the annual demand is automatically reflected on the lead-time demand, the lead-time plus one period's demand, the shortage etc., assuming the annual demand to be a fuzzy random variable automatically implies that these parameters are fuzzy random variables as well. In the subsequent section, the means of realizing the motive behind the work done has been outlined.

1.6 Our approach

The aim of the thesis has been realized by developing some of the most fundamental models of inventory control under the fuzzy random framework.

One of the most widely employed and fundamental inventory models that are employed in real life are the review systems. In the review systems, the issues to be dealt with are as follows:

(i) ***When should a replenishment order be placed?***

That is whether the order should be placed at the time of review or should it placed when the inventory level falls to a particular level.

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(ii) ***What should be the size of the replenishment order?***

That is whether the order should be a fixed quantity every time an order is placed or should it vary so as to reach a fixed target.

Based on the above issues, the review systems can be broadly classified into two types – the continuous review system and the periodic review system. In the continuous review inventory control system, the inventory position is monitored continuously. On the other hand, in periodic review inventory control system, the inventory position is monitored at regular intervals. The main difference between the continuous review system and the periodic review system is that in the continuous review system, the order size is fixed while the time between successive orders varies. In the periodic review system, the time between consecutive orders is fixed while the order quantity varies. There are a number of variations and combinations of these two policies as well. But these two form the basis of the other review systems. Thus these two fundamental inventory control systems have been focused on in the thesis. The main features of the continuous and periodic review systems have been discussed briefly in the next sub section.

Another type of inventory model that finds wide application in real life is the single period problem. Products with very short selling seasons, sports apparels, greeting cards etc are some of the examples where single period models are employed. This problem, also referred to as the 'newsboy' problem has also been developed in the fuzzy random environment in this thesis and has also been discussed in brief in the following section.

1.6.1 Continuous review system

One of the most commonly used continuous review policy is the (Q, r) operating policy (Hadley and Whitin 1963). In this policy, a quantity Q is ordered (or produced, in case of called the reorder point r. The purpose of the analysis is to determine the optimal values of the order quantity Q^* and the reorder point r^* such that the total cost in minimized.

Quite often it is seen that there is a time lag between the placement of an order and its actual arrival. This is defined as the lead-time L.

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The total inventory cost for the continuous review system can be calculated as follows:

Total annual inventory cost = annual ordering cost + annual carrying cost + total expected shortage cost

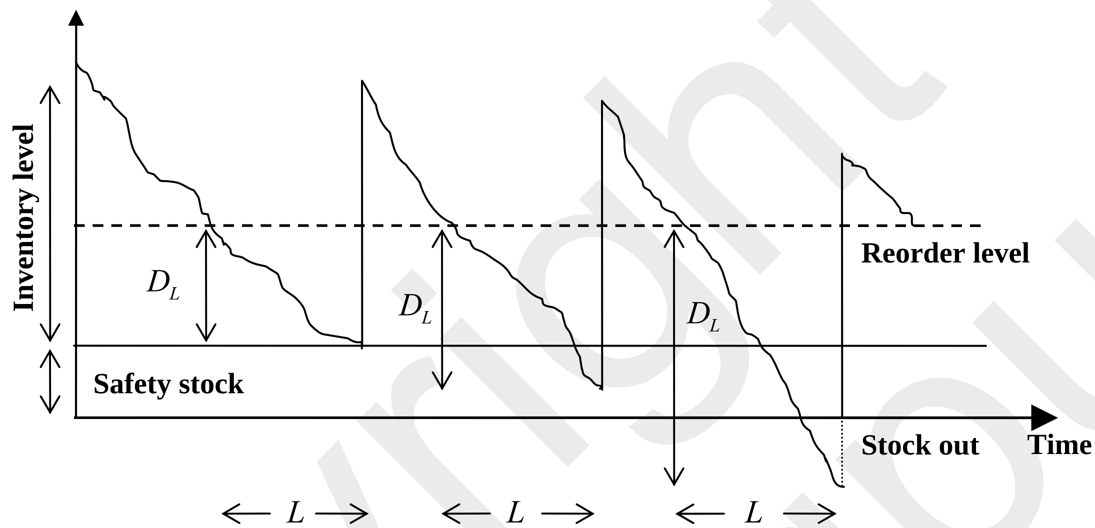

Fig 1.2 Continuous review system

Now in case of a stock out situation, one of two cases may happen – the customers may be willing to wait to receive their orders with the next arrival of stock or they may take their orders elsewhere. As the nature of the two cases imply, the first is referred to as the backorders case while the second is lost sales.

For the backorder case, the total cost is given by

$$C(Q,r) = (D/Q)P + h\{Q/2 + r - \mathrm{E}(D_L)\} + (D/Q)\pi\bar{b}(r)$$

And for the lost sales the total cost is

$$C(Q,r) = (D/Q)P + h\{Q/2 + r - \mathrm{E}(D_L)\} + \{h + (D/Q)\pi\}\bar{b}(r),$$

where Q is the order quantity, r is the reorder point, P is the setup cost (may be replaced with the cost of placing an order C_o for external source of replenishment), h is the inventory holding cost per year, π is the shortage or penalty cost per unit shortage, D is the annual average demand, D_L is the random lead-time demand, $\mathrm{E}(D_L)$ is the expectation of the lead-time demand, $\bar{b}(r)$ is the expected shortage per cycle, which is given by $\bar{b}(r) = \int_r^\infty (D_L - r) f(D_L)\, dD_L$, where $f(D_L)$ is the probability density function of the lead-time demand. It is to be noted that the expected number of backorders per cycle is the same as the expected number of lost sales per cycle.

The safety stock SS is defined as the expected net inventory just prior to the arrival of an order. So for the backorder case, the safety or buffer stock is defined as

$$SS = r - \mathrm{E}(D_L)$$

For the lost sales case, the safety stock expression is modified as

$$SS = r - \mathrm{E}(D_L) + \int_r^\infty (D_L - r) f(D_L)\, dD_L$$

For the backorders case, the safety stock may be positive, zero or even negative while for the lost sales, it should always be non-negative. However, in reality, when it is expensive to incur backorder or lost sales cost, then it is mostly cheaper to carry some additional stock. Besides, frequent stockout situation may also lead to loss of goodwill on part of the decision maker and as such a non-negative criterion has been imposed on the safety stock for both the continuous review models discussed in the thesis.

1.6.2 Periodic review system

The most popular and widely used periodic review operating policy is what is known as the 'order up to R' doctrine, also referred to as the periodic review (R,T) system.

In this doctrine, the inventory level is monitored at regular intervals and at the time of review, an order is placed to bring the inventory position i.e., the amount on hand plus on order up to a certain level R. The purpose of the analysis is to determine the optimal

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values of the review period T^* and the target inventory level R^* such that the total cost in minimized.

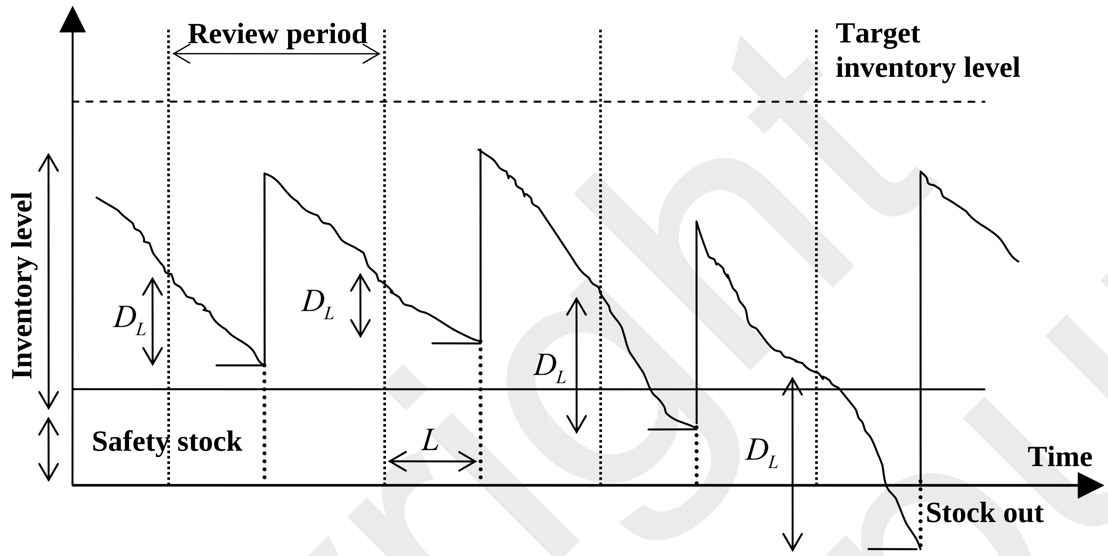

Fig 1.3 Periodic review system

For the periodic review system, the total cost is calculated as follows:

Total annual inventory cost = annual ordering cost + annual carrying cost + total expected shortage cost

For the backorder case the total cost is given as

$$C(R,T)=\frac{A}{T}+h\left[R-\mathrm{E}(D_L)-\frac{DT}{2}\right]+\frac{\pi}{T}\bar{b}\left(D_{L+T}-R\right)^+$$

And for the lost sales case it is

$$C(R,T)=\frac{A}{T}+h\left[R-\mathrm{E}(D_L)-\frac{DT}{2}\right]+\left(h+\frac{\pi}{T}\right)\bar{b}\left(D_{L+T}-R\right)^+$$

where T is the time between reviews, R is the target inventory level, J is the cost of making a review, C_o is the cost of placing an order $(A=J+C_o)$, π is the unit backorder

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cost, h is the unit holding cost, D is the annual average demand, L is the constant lead-time, D_L is the random lead-time demand, D_{L+T} is the random lead-time plus one period's demand, $\mathrm{E}\left(D_L\right)$ is the expected lead-time demand, $\overline{b}\left(D_{L+T}-R\right)^{+}$ is the expected shortage and it is given by $\overline{b}\left(D_{L+T}-R\right)^{+}=\int_{R}^{\infty}\left(D_{L+T}-R\right)f\left(D_{L+T}\right)dD_{L+T}$, where $f\left(D_{L+T}\right)$ is the probability density function of the lead-time plus one period's demand.

The safety or buffer stock for the backorder case is

$$SS = R - \mathrm{E}\left(D_L\right) - DT$$

For the lost sales case, the expression for safety stock is modified as

$$SS = R - \mathrm{E}\left(D_L\right) - DT - \int_{R}^{\infty}\left(D_{L+T}-R\right)f(D_{L+T})dD_{L+T}\,.$$

As mentioned earlier, the safety stock has been maintained at a non-negative level for both the periodic review systems discussed in the thesis.

1.6.3 Single period inventory model

Single period inventory model is one of the most commonly found inventory models in the real world. In the single period problem, there is only a single replenishment opportunity at the start of the selling season. Since this feature is commonly found for newspapers so, this type of single period problem is popularly referred to as the 'newsboy' problem. If the quantity ordered is greater than the demand, then the cost is incurred in terms of inventory being kept in stock. On the other hand, if the order quantity is less than the demand, then profit is lost. The main objective then is to determine the optimum trade-off between overstocking (inventory holding cost) and understocking (shortage or penalty cost). This leads to the principal aim of the problem which is to determine the optimal order quantity such that the total cost is minimized or the total profit is maximized. Business dealing with products like Christmas trees, greeting cards, fashion products, sports goods and apparel etc. are suitable examples where newsboy

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problem finds a wide application. Generally, the single period inventory model is represented as follows:

$$\mathrm{P} = \begin{cases} pD - cQ - h(Q-D), \; D \le Q \\ (p-c)Q - \pi(D-Q), \; D \ge Q \end{cases}$$

where P is the profit and p, c, h, π are the selling price, the cost price, the holding cost and the shortage cost of unit item, respectively. The costs are all assumed to be independent of time. The demand is denoted by D while Q is the quantity procured. So the aim of the decision maker is to determine the optimal order quantity Q^* such that the total profit is maximized.

1.7 Objective of the thesis

The objective of the thesis is therefore to explore the applicability and effect of assuming the annual customer demand to be a discrete or continuous fuzzy random variable in the field of inventory management by developing the following models in the fuzzy random framework:

- the standard periodic review system following an 'order up to R' doctrine
- the continuous review (Q, r) system with a mixture of backorders and lost sales and a restriction on the total available budget introduced as an 'imprecise' chance constraint
- the periodic review system with variable lead-time and a negative exponential lead-time crashing cost
- a single period inventory model with resalable returns under fuzzy costs and with the associated probabilities of the demand assumed to be fuzzy as well
- a single period inventory model with fuzzy random variable demand following a continuous probability distribution
- a continuous review inventory system with setup cost reduction and quality improvement and with continuous fuzzy random variable demand following a uniform distribution

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With these above objectives, the thesis has been organized in the following manner.

1.8 Organization of the thesis

As mentioned in the section above, the aim of this thesis is to investigate the effect and applicability of integrating the two concepts of fuzziness and randomness in the field of inventory management. And this has been accomplished by developing two of the most fundamental inventory models – the continuous and periodic review systems and the single period 'newsboy' problem – with various assumptions under the fuzzy random framework. The thesis has been organized to this effect as follows:

Chapter 1 of the thesis is the introductory chapter where the essential concepts of inventory management problems have been presented and the motivation for the work done has been outlined. The preliminary concepts that have been used for the analysis of the models have also been discussed in brief.

In Chapter 2, a periodic review inventory system following an 'order up to R' doctrine has been analyzed with the annual customer demand assumed to be a discrete fuzzy random variable. The lead-time has been assumed to be a constant while the lead-time demand and the lead-time plus one period's demand have been connected to the annual demand by means of the lengths of the lead-time and period. A methodology has been developed to determine the optimal target inventory level and the optimal period of review such that the total annual cost is minimized. An algorithm has been presented to encapsulate the methodology and it has been illustrated by way of a numerical example. With the help of the example it has been shown that the result obtained for the fuzzy random model follows the same trend as the established probabilistic periodic review models. An illustration of the application of the proposed model to a real life inventory situation has also been presented.

In Chapter 3, a continuous review (Q, r) inventory system with a mixture of backorders and lost sales with imprecise chance budget constraint has been considered. This model presents a possible method of quantifying 'fuzzily defined uncertain' information which

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is most readily available in any real life inventory situation. As in the previous chapter, the model has been analyzed with the customer demand incorporated as a discrete fuzzy random variable and the lead-time demand has been connected to the annual demand through the length of the lead-time. A methodology has been proposed to determine the optimal order quantity and the reorder point such that the total cost incurred is minimized subject to the constraints. A numerical example has been presented to illustrate the method.

Chapter 4 extends the fuzzy random periodic review model of Chapter 2 by including the effect of controlling the lead-time and adding a crashing cost, as a negative exponential function of the lead-time, to that effect. A methodology has been developed such that the total cost is minimized and the optimal period of review, the optimal target inventory level and also the optimal lead-time are determined in the process. Through the numerical example it has been illustrated that considering lead-time crashing cost leads to a lower safety stock level as expected. Further reduced lead-time also provides the company an added advantage in terms of providing its customers with a more speedy delivery of goods.

A single period inventory model with resalable returns has been analyzed in Chapter 5 with the annual customer demand assumed to be a discrete fuzzy random variable with imprecise probabilities. This model investigates the phenomenon of 'resalable returns' which is widely found in internet or mail order companies dealing with single period items like highly seasonal products, fashion and style goods etc. As these businesses are characterized by lack of historical data, the probabilities that a sold product is returned and that a returned product is resalable have also been assumed to be fuzzy in nature. For computational purpose, the total profit has been defuzzified using 'modified' graded mean integration representation. This 'modified' graded mean integration representation of a fuzzy number has been developed in this chapter by incorporating the attitude of the decision maker. The profit has been maximized and the optimal order quantity determined in the process. A numerical example illustrates the methodology.

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In Chapter 6, a single period inventory model has been presented with the customer demand assumed to be a continuous fuzzy random variable which follows some probability distribution. The methodology of the model is developed first for the somewhat simple uniform distribution and then for the most commonly used normal distribution. Since assuming demand to follow a normal distribution implies that some negative demand is automatically being considered, the case for the normal distribution has been developed using a suitable modification of it, if required by the situation. The model has been developed with the aim of realizing a target profit and this issue has been formulated as a fuzzy inequality. A methodology has been developed based on fuzzy inequality relation. Through numerical examples, the computational procedure has been illustrated. It has also been shown that, for certain conditions, using the left truncated normal distribution left truncated at zero, in place of the normal distribution may prove to be beneficial for the decision maker in terms of higher profit and lower order quantity.

A fuzzy random continuous review (Q,r) inventory system has been developed in Chapter 7 with the annual customer demand assumed to be a continuous fuzzy random variable following uniform distribution. The effect of investments made to reduce the setup cost and improve the quality of the products has been investigated here by considering them to be additional control parameters. The total cost has been minimized and the optimal solution obtained. By means of numerical examples it has been shown that considering the investments made to reduce setup cost and improve product quality leads to smaller lot sizes of better quality products which in turn lead to the successful implementation of the Just-In-Time (JIT) manufacturing philosophy.

Finally the concluding remarks regarding the work carried out and some scope of carrying out future work related to these are presented in Chapter 8.

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