Chapter 1

Introduction

The computation of fixed points/solutions of nonlinear operator equations in Banach spaces is a challenging problem in numerical analysis. Many applications such as Kinetic theory of gases, elasticity and of other applied areas are generally reduced to solving nonlinear equations depending on one or more parameters. Dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the equilibrium states of the systems obtained by solving these equations. Many optimization problems also lead to solutions of these equations. As a result many methods of various orders are developed for solving them. Generally, iterative methods are used for the solutions of these equations. All these methods satisfy a number of criteria such as they should enjoy good convergence properties, be efficient and be numerically stable. For a good review of these methods, one may refer the excellent books of Traub (1964), Ostrowski (1966), Rall (1969), Ortega and Rheinboldt (1970), Kantorovich and Akilov (1982) and Argyros and Szidarovszky (1993). In addition to solving nonlinear equations, another problem which has gained attention of many researchers is that of enclosing their solutions in $\mathbb{R}$. It is evident that the globally convergent bisection, regula falsi and other similar methods generate sequences of intervals enclosing the roots and converge from two sides to these roots in the limiting case. However, the rate of convergence of these methods are either linear or superlinear. The quadratically convergent Newton-Fourier method generates two sequences for strictly increasing and convex functions that are monotonically convergent from two sides to the root. Ostrowski (1973, p.70) showed that the sequence of diameters of these intervals is Q-quadratically convergent to 0. By considering the regula falsi and bisection methods as auxiliary methods, Alefeld and Potra (1992) proposed three new methods which produce sequences of nested intervals without any convexity assumptions. The first method requires asymptotically two function values per step. The diameters of the enclosing intervals are Q-quadratically convergent to 0 so that the efficiency index of the method is $\sqrt{2} = 1.4142\ldots$. The second method requires asymptotically three function values per step and the Q-convergence order of the sequence of diameters is four. Hence its efficiency index is $4^{1/3} = 1.5874\ldots$. The third method uses three function values at each step and has R-order convergence $(3+\sqrt{13})/2 = 3.3027\ldots$. The corresponding efficiency index is $1.4892\ldots$. By considering the Bernoulli method instead of the regula falsi and bisection methods as auxiliary methods, Costabile et al. (1999, 2001) gave three algorithms that improved the results of Alefeld and Potra (1992). But all these methods are either the derivative free or contain the derivatives. The derivative free methods converge slowly in comparison with the methods having derivatives. Wu and Fu (2001) proposed a method for finding the roots of nonlinear equations in $\mathbb{R}$ which is a combination of the bisection method and a derivative free Newton-like method. They showed that starting with the initial interval $[a_0, b_0]$ containing the zero x^*, the sequence of diameters $\{(b_n - a_n)\}$ and the sequence of errors $\{(x_n - x^*)\}$ generated by the method converges quadratically to 0. Zhu and Wu (2003) later modified it and showed that, both the sequence of diameters and the sequence of errors are cubically convergent to 0. Wu et al. (2003) gave an improved regula falsi method in which by combining the regula falsi method with the Steffensen's method they showed that both the sequence of diameters and the sequence of errors are quadratically convergent to 0 simultaneously. Chen and Li (2007) have developed a class of quadratically convergent exponential iterative methods. These methods are then combined with classical regula falsi method to establish that both the sequence of diameters and the sequence of errors produced by these methods asymptotically converge to 0. Recently a number of higher order methods are also appearing to solve nonlinear equations in $\mathbb{R}$. Neta (1983) developed a family of sixth order methods which requires evaluation of three functions and one first derivative per iteration. Sharma and Guha (2007) developed a one parameter family of sixth order methods based on Ostrowski fourth order multipoint method. Chun (2005) presented a one parameter family of variants of Jarratt's fourth order method for solving nonlinear equations. It is shown there that the order of convergence of each family member is improved from four to six even though it adds one evaluation of the function at the point iterated by Jarratt's method per iteration. Kou and Wang (2007) presented a family of new variants of the Chebyshev-Halley methods having sixth order of convergence although they only add one evaluation of the function at the point iterated by the Chebyshev-Halley method. It is to be noted that most of the methods in $\mathbb{R}$ can be extended to Banach spaces, by writing inverse operators instead of their quotients. The quadratically convergent Newton's method and its variants are well known and most widely used efficient iterative methods for solving nonlinear equations in Banach spaces. The Kantorovich theorem (see Kantorovich and Akilov (1982)) gives sufficient conditions for the semilocal convergence, the error estimates and the existence-uniqueness regions of the solutions for Newton's method. Basically, two different kinds of proof are given by Kantorovich. The first one is based on the recurrence relations which supposes some advantages, because one can reduce the initial problem in a Banach space to a simpler problem with real sequences and vectors. Besides, we find a symmetry between some special properties of the iteration method and the corresponding ones in the system of recurrence relations. Other one is based on the majorizing principle, where the sequence of iterates generated by the iterative method in Banach space is majorized by the sequence of iterates generated by the same iterative method applied on a scalar function. Both the cases provide error estimates for the method. Others have modified these proofs to get optimum error estimates.

The main assumption of Kantorovich's theorem is the second Fréchet derivative of the involved operator be bounded in some ball around the initial point. Others have also established a number of convergence theorems for their methods based on the different sufficient conditions on the second derivative of the involved operator. The main practical difficulty in all these is the evaluation of the second derivative which requires huge amount of computation or even may not exist sometimes. To avoid this, many researchers replaced the second derivative by different continuity conditions on the first Fréchet derivative of the operator. Yamamoto (1985, 1986a,b) gave a priori and a posteriori error bounds for Newton's method by using the recurrence relations under an affine invariant Lipschitz continuity condition on the first derivative of the operator. Rall (1975) established the convergence of Stirling's method under the assumption that the first derivative satisfies the Lipschitz continuity condition by using the recurrence relations. This method can be viewed as a combination of the method of successive substitutions and Newton's method for a real function. It is comparable with the well known quadratically convergent Newton's method. In terms of computational effort both these methods require essentially the same labor per step. However, the main advantage of Stirling's method is that it can obtain a fixed point in cases where Newton's method fails. Gutiérrez and Hernández (2000) weaken the Lipschitz continuity condition by fixing one point to the initial point. It is observed that the first derivative of the nonlinear operators such as the nonlinear Hammerstein-type integral equations of the second kind (see Hernández (2001b)) or the systems of nonlinear equations obtained from the nonlinear boundary value problems (see Rokne (1972)) fails to satisfy the Lipschitz condition but satisfies the more general Hölder continuity condition. This implies that the Hölder continuity condition is weaker than the Lipschitz continuity condition. Thus, the convergence of Newton's method is analyzed when the first derivative of the involved operator is Hölder continuous. Hernández (2001c) and Ezquerro and Hernández (2002a,b, 2006) proposed further weaker condition termed as the ω-condition on the first/second derivative of the operator to establish it's convergence and showed that the Lipschitz and Hölder conditions on the first derivative and the boundedness condition on the second derivative of the operator are special cases of this condition. A priori error bounds are also obtained under these relaxed conditions for Newton's method. Sometimes, we need a quick convergence of the iterative methods used for solving operator equations in Banach spaces. This motivates the development of higher order iterative methods. It is clear that the higher the order of the method, the higher will be the rate of convergence. However, the operational cost of these methods also increases with the order. This fact leads to find an equilibrium between the high rate of convergence and the operational cost. The situation that Newton's method and the similar second order methods seem to attain equilibrium has lead to wrong idea that the higher order methods have no more than theoretical interest. Of course, the higher order methods require more computational cost than other simpler methods which make them disadvantageous to use in general. But in some cases they can be applied in practice. This assertion is confirmed by the big number of publications about these methods. These methods are important because many applications such as stiff systems of equations require quick convergence of their solution methods. Third order methods are also good alternatives to Newton-type methods in finding the solutions of the quadratic equations in Banach spaces when the second derivative is a constant bilinear operator. Some particular cases of these types of equations which appear in many applications, such as control theory, are Riccati's equations (see Lancaster and Rodman (1995)). Besides, these methods are also interesting from the theoretical stand point because they provide the results on the existence and uniqueness of the solution that improve the results given by Newton's method. There exists a large number of methods which accelerate the order of convergence from quadratic to cubic. Argyros and Chen (1994) established the semilocal convergence of the midpoint method under Ptâk-like assumption on the first Fréchet derivative of the operator. Ezquerro et al. (1998) and Ezquerro and Hernández (2004) used the Lipschitz continuity condition on the first derivative to study the convergence of the midpoint method and the Halley's method, respectively. Argyros (1993a) and Ezquerro et al. (2002b) established the convergence of Newton-like methods under the Hölder continuity condition on the first Fréchet derivative of the involved operator. Ye et al. (2007) established the semilocal convergence of the variants of the Chebyshev-Halley's family of iterative methods with parameters for solving nonlinear operator equations in Banach spaces under the Hölder continuity condition on the first Fréchet derivative.

The thesis is organized as follows: It consists of nine chapters of which Chapter 1 is the introduction. Chapter 2 gives a brief review of the literature survey on solving nonlinear equations in Banach spaces.

In Chapter 3, a class of derivative free third order exponential iterative methods is developed for finding roots of the nonlinear equations in $\mathbb{R}$. This is done by introducing a function suitably defined in the iteration of Chen and Li (2007). The new class of methods is then combined with the regula falsi method to establish that both the sequence of diameters and the sequence of iterates produced asymptotically converge to 0 simultaneously. The numerical examples are also worked out to show that our modified methods are more effective and comparable to those given by Chen and Li (2007) as well as Newton's method, Steffensen's method and regula falsi method.

In Chapter 4, we have developed a three steps sixth order method by combining the third order method of Weerakoon and Fernando (2000) with Newton's method to solve the nonlinear equations in $\mathbb{R}$. In terms of the computational cost, it requires the evaluation of only two functions and two first derivatives per iteration. This gives the efficiency index equal to 1.565... which is better than Newton's method and the method of Weerakoon and Fernando. The efficacy of the method is tested on a number of numerical examples. It is observed that our method takes less number of iterations than Newton's method and the method of Weerakoon and Fernando. On comparison with the other sixth order methods, it behaves either similarly or better for the examples considered.

In Chapter 5, using the recurrence relations, the semilocal convergence of Stirling's method used to find fixed points of the nonlinear equations in Banach spaces is established under the assumption that the first Fréchet derivative satisfies the Hölder and weak Hölder continuity conditions. These continuity conditions are milder than the usual Lipschitz continuity condition. The importance of our work lies in the fact that there exists many problems where the Lipschitz continuity condition fails but the Hölder continuity condition holds on the first Fréchet derivative of the involved operator. Convergence theorems are established to derive a priori error bounds along with the existence and uniqueness regions for the fixed points. A number of numerical examples are worked out to show the efficacy of our approach.

The Chapter 6 further relaxes the Hölder and weak Hölder continuity conditions of Chapter 5 by ω- and weak ω- continuity conditions for finding fixed points of the nonlinear operator equations in Banach spaces. Examples are found in which the first Fréchet derivative fails to satisfy both the Lipschitz and Hölder continuity conditions but satisfies ω-continuity condition. Using the recurrence relations, the convergence theorems are established to obtain a priori error bounds along with the existence and uniqueness region for the fixed points. Numerical examples are also worked out to demonstrate the efficacy of our approach.

In Chapter 7, the semilocal convergence of a third order Stirling-like method used to find fixed points of the nonlinear equations in Banach spaces is established by using the recurrence relations under the assumption that the first Fréchet derivative of the involved operator satisfies the Lipschitz continuity condition. A convergence theorem is established to derive a priori error bounds along with domains of the existence and uniqueness for a fixed point. The R-order of the method is shown to be equal to 3. Two numerical examples are worked out and the results obtained are compared with a third order Newton-like method. It is found that our results are computationally more efficient.

The Chapter 8 relaxes the Lipshitz continuity condition of Chapter 7 by the Hölder continuity condition for a Stirling-like method used to find fixed points of the nonlinear operator equations in Banach spaces. A family of recurrence relations based on a parameter depending on the involved operator is used to derive a priori error bounds. The R-order of convergence of the method is shown to be equal to $(2p+1)$ for $p \in (0,1]$ by using a new family of recurrence relations. A number of numerical examples are worked out and the results obtained are compared with those obtained by some other existing third order methods. It is observed that our approach leads to better error bounds and the existence and uniqueness regions for a fixed point.

The Chapter 9 uses the ω-continuity condition for a Stirling-like method used to find fixed points of nonlinear operator equations in Banach spaces. A new family of recurrence relations are derived based on a parameter that depend on the involved operator. A convergence theorem is given to derive a priori error bounds along with the domains of existence and uniqueness of a fixed point. The R-order of the method is shown to be equal to $(2p+1)$ for $p \in (0,1]$. Two numerical examples are given to demonstrate the applicability of our approach.

Finally, the conclusions and scope of future works are given.