Abstract

There are a number of graphs associated with semigroups and groups. The Cayley graph is the oldest and most popular among them whereas the concept of power graph is a recent development. In this thesis our main subject of study is the power graph. If G is a finite semigroup or group then the power graph of G, denoted by $\mathcal{G}(G)$, is an undirected graph with vertex set G and two vertices $u, v \in G$ are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m.

Investigation of various graph theoretic properties of power graphs, determination of adjacency and Laplacian spectrum of them, and finding connection between power graphs and Cayley graphs are the main concern of the thesis. We give necessary and sufficient conditions for planarity of power graphs of finite cyclic group H_n (or $\mathbb{Z}_n$, both the notations are used in the thesis as and when required), dihedral groups D_{2n} and dicyclic groups (or generalized quaternion groups) Q_{4n}. We find bounds for vertex-connectivity of $\mathcal{G}(H_n)$ and prove that $\mathcal{G}(D_{2n})$ is 1-connected whereas $\mathcal{G}(Q_{4n})$ is 2-connected. We also prove that both the graphs $\mathcal{G}(D_{2n})$ and $\mathcal{G}(Q_{4n})$ are neither Hamiltonian nor Eulerian. After that we concentrate on adjacency and Laplacian spectrum of the power graphs of finite cyclic, dihedral and dicyclic groups. We give formulae for adjacency (resp. Laplacian) characteristic polynomial of the power graph of finite cyclic groups H_n (resp. $\mathbb{Z}_n$) in terms of a suitable minor of adjacency matrix (resp. Laplacian matrix) of $\mathcal{G}(H_n)$ (resp. $\mathcal{G}(\mathbb{Z}_n)$) formed by non-identity and non-generator elements. We also find the adjacency and Laplacian characteristic polynomial of $\mathcal{G}(D_{2n})$ in terms of that of $\mathcal{G}(H_n)$ and find the full spectrum of power graphs of generalized quaternion 2-groups. Further we give bounds for spectral radius of $\mathcal{G}(H_n)$, $\mathcal{G}(D_{2n})$ and $\mathcal{G}(Q_{4n})$. Using vertex connectivity we obtain bounds for algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$ and find that the algebraic connectivity of $\mathcal{G}(D_{2n})$ is 1 whereas that of $\mathcal{G}(Q_{4n})$ is 2.

Next we are interested on the subgraph $\mathcal{G}(\mathbb{Z}_n) - Y_n$, where Y_n is the subset of $\mathbb{Z}_n$ consisting of the identity and all generators. We study some graph theoretic and spectral properties of the graph $\mathcal{G}(\mathbb{Z}_n) - Y_n$ as it helps us to study some properties of $\mathcal{G}(\mathbb{Z}_n)$ as well as of $\mathcal{G}(D_{2n})$ more precisely. Necessary and sufficient conditions are given for $\mathcal{G}(\mathbb{Z}_n) - Y_n$ to be connected, planar or a complete graph. We find some adjacency and Laplacian eigenvalues of $\mathcal{G}(\mathbb{Z}_n) - Y_n$. For a group G and any positive integer k with $k \geq 2$, we define a spanning subgraph $\mathcal{G}(G, k)$ of the power graph $\mathcal{G}(G)$, called k-power graph, where two distinct vertices u, v are adjacent if and only if $u^k = v$ or $v^k = u$. We study some structural properties of k-power graphs of the groups H_n, D_{2n} and Q_{4n}.

Finally we connect power graphs with Cayley graphs. We prove that the vertex deleted subgraphs of some power graphs are spanning subgraphs or equal to the complement of vertex deleted subgraphs of some unitary Cayley graphs. Also we show that some Cayley graphs can be expressed as direct product of power graphs. Applying these relations we study the eigenvalues and energy of power graphs and the related Cayley graphs.

Keywords: Graph; Subgraph; Spanning tree; Vertex connectivity; Planarity; Eulerian graph; Hamiltonian graph; Finite group; Cyclic group; Dihedral group; Dicyclic group; Cayley graph; Unitary Cayley graph; Power graph; Iteration digraph; Adjacency and Laplacian Matrix; Characteristic polynomial; Spectrum; Spectral radius; Algebraic connectivity; Graph Energy.