SYNOPSIS

One of the most common problems in Numerical Analysis is that of computing a function $f(x)$ for one or more given values of its argument x. The function may be defined as a series, a definite integral, as the solution of a differential or integral equation or some other computable form. The present thesis thus concerns with the study of two types of problems, namely, that of evaluating an integral over semi-infinite and infinite intervals with exponential weight functions and of generating polynomial and non-polynomial approximation of functions. The work has been divided into seven chapters.

Chapter I presents some of the appealing results of researches in Chebyshev and Gaussian quadrature and in approximation of functions. Chapters II, III and IV are concerned with the problem of finding a numerical method to evaluate an integral. Amongst many numerical methods to evaluate an integral, the methods of Chebyshev, Gauss and Newton-Cotes are the basic ones and the most commonly used are Gaussian and Newton-Cotes type. Thus, Chapter II presents the problem of examining, not so commonly used, Chebyshev quadrature for integrals of the type $\int_a^\infty \exp(-x^m)\, f(x)\, dx$ when $m = 2p-1$, $a = 0$ and $m = 2p$, $a = -\infty$ and also of finding the possibility of a weight function and an interval so that n-point Chebyshev

quadrature may have real nodes for all values of n and hence may be of use to evaluate integrals. Numerical computations show that the range of existence of real Chebyshev quadrature increases with the increase in p and tends to the case of Chebyshev quadrature for $\int_0^1 f(x)\,dx$ and $\int_{-1}^1 f(x)\,dx$ when $p = 64$ and 27 respectively. Furthermore, this leads us to state and prove that n-point Chebyshev quadrature for $\int_{-1}^{1} \frac{\exp(-x^{2p})}{\sqrt{1-x^2}} f(x)\,dx$ will have real nodes for nearly every value of n whenever $p >$ some p_0. A technique by which Chebyshev quadrature with complex nodes can be used profitably to evaluate integrals is also given.

The type of problem discussed in Chapter II and the problem of evaluating $\int_0^\infty e^{-x^4} \cos x\,dx$ numerically led us to generate the weights and abscissas of Gaussian quadrature for integrals of the type $\int_a^\infty \exp(-x^m)\, f(x)\,dx$ for two cases : (i) $m = 2p-1$, $a = 0$ and (ii) $m = 2p$, $a = -\infty$ and the results are presented in Chapter III. The tables enable us to evaluate many integrals which could not be evaluated by the existing tables on Gaussian quadrature. Chapter IV deals with employing variational method to find an approximation of $\int_x^b f(x)\,dx$ where $f(x)$ is any twice continuously differentiable function in $a \leqslant x \leqslant b$. It has been shown that the integral $\int_x^b f(x)\,dx$ could be approximated by

$$\frac{(f(x))^{2n}}{\dfrac{E_x(V)}{V^2(x)} - nf'(x)(f(x))^{2n-2}}$$

where

(i) $$E_x(V) = \int_x^b \left[(f(t))^{2n-1} V'^2 + n(f(t))^{2n-3} \times \left\{ (2-n)f'^2 - ff'' \right\} V^2 \right] dt$$

is positive definite,

(ii) $V(x)$ has a piecewise continuous first derivative in $a \leqslant x \leqslant b$ with $V(b) = 0$,

(iii) $\underset{x \to b}{\mathrm{Lt}} \ (f(x))^{-n} \int_x^b f(t)\, dt \to 0$, and

(iv) n is a convenient real number subject to the restrictions (i) and (iii) mentioned above.

Chapters V and VI centre round the problem of finding polynomial and non-polynomial approximation of functions.

If $p_n(x)$, a polynomial of degree n, is an approximation to a continuous function $f(x)$ and $e_n(x) = f(x)-p_n(x)$ is the error of approximation then the problem is to determine the polynomial $p_n(x)$ so that the error $e_n(x)$ is minimum in some sense. Accordingly, Chapter V concerns with generating polynomial approximation of the identically vanishing function with exponential weight function, $\exp(-x^m)$, in (a,b) for two cases : (i) $m = 2r-1$, $(a,b) = (0, \infty)$ and (ii) $m = 2r$, $(a,b) = (-\infty, \infty)$ by using the L_p and minimax criteria of minimizing the error e_n. The numerical results show that for the problem under consideration, L_p approximations do not tend to minimax approximations as $p \to \infty$, a result quite contrary to the classical result that L_p approximations tend to minimax approximations over finite intervals.

Chapter VI discusses the problem of determining the approximation of a function, $f(x)$, as sum of exponentials by using the technique of differential approximation. It has been shown with the help of examples that the maximum error in the approximation $V(x,a,\lambda) = \sum_{i=1}^{n} a_i e^{\lambda_i x}$ to $f(x)$ obtained by determining $b_1, b_2, \ldots, b_n$ so that

$$(1) \qquad \int_0^1 \left| f^{(n)} + b_1 f^{(n-1)} + \ldots + b_n f \right|^2 dx$$

is minimum with respect to $b_1, b_2, \ldots, b_n$ and then solving

$$V^{(n)} + b_1 V^{(n-1)} + \ldots + b_n V = 0$$

so that

$$(2) \qquad \max_{0 \leq x \leq 1} \left| f(x) - V(x,a,\lambda) \right|$$

is minimum with respect to $a_1, a_2, \ldots, a_n$ is much smaller than for the approximations obtained by replacing (1) by minimax norm or (2) by initial conditions $V^{(i-1)}(0) = f^{(i-1)}(0)$, $i = 1,2, \ldots, n$. Furthermore, this approximation is such that the error curve $f(x) - \sum_{i=1}^{n} a_i e^{\lambda_i x}$ alternates exactly n times on $(0,1)$.

Finally, Chapter VII consists of discussing iteration functions of the type

$$\phi = x - \frac{\sum_{i=1}^{m} a_i f(x - \beta_i \frac{f(x)}{f'(x)})}{\sum_{i=1}^{n} A_i f'(x - B_i \frac{f(x)}{f'(x)})}$$

for various values of m and n. It has been found that this type of iteration function can be at most of order 4 except for $m = 2$, $n = 1$; and $m = 1$ and any n in which case the iteration function is of order 3. For $m = 1$ and any $n > 1$ it is always possible to have a family of iteration functions of order 3 with the asymptotic error constant $\left(\frac{f''}{2f'}\right)^2$.