CHAPTER - I

INTRODUCTION

1. Preliminary Remarks and Governing Equations

The study of the dynamics of fluids in rotating systems constitutes an important branch of theoretical fluid mechanics and has developed rapidly in recent years. The rotation gives rise to a range of new phenomena which are important in a wide variety of problems - from the flow in a biochemist's centrifuge to the large-scale circulations in the atmosphere and oceans.

The hydrodynamic phenomena which occur in rotating fluids can be conveniently studied in a rotating frame of reference. The equations governing the motion of an incompressible viscous fluid in a coordinate system rotating with angular velocity $\vec{\Omega} = \Omega \vec{k}$ are [1]

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} = -\frac{1}{\rho} \nabla P - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2 \vec{\Omega} \times \vec{q} + \nu \nabla^2 \vec{q} \qquad (1)$$

$$\nabla \cdot \vec{q} = 0 \qquad (2)$$

Here $\vec{q}$ is the particle velocity in the rotating frame and $\vec{r}$, t, P, ρ, ν are respectively, the position vector, time, pressure, density and kinematic viscosity. The second and third terms on the right hand side of (1) are respectively the centrifugal and coriolis forces. The centrifugal term can be combined with P because it can be expressed as the gradient of a scalar,

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\nabla\left(\frac{1}{2}\Omega^2 r'^2\right),$$

where r' is the distance from the axis of rotation. Thus the first two terms on the right hand side of (1) can be replaced by $-\frac{1}{\rho}\nabla p$ where $p = P - \frac{1}{2}\rho\Omega^2 r'^2$.

Equations (1) and (2) are to be solved satisfying appropriate boundary and initial conditions.

2. Non-dimensionalization scheme

Ideally, the set of equations (1) and (2) together with the boundary conditions constitute a well-posed system for which a solution should be obtainable. Practically, however, this solution is difficult to obtain. It is thus necessary to resort to a wide variety of ingenious methods by which some simplifications may be introduced. The technique of non-dimensional analysis and ordering is one such method. In addition to pointing out the parameters influencing the flow, the non-dimensionalization of a differential system highlights the importance of certain physical effects, relative to others. It thus becomes possible to have some mathematical simplification by neglecting various terms which are considered to be relatively small in comparison with other terms.

If L , t_o and U characterize the length, time and relative velocity of any phenomenon under study the replacement of variables by their scaled counterparts reduces the equations (1) and (2) to

$$\frac{1}{\Omega t_o}\frac{\partial \vec{q}^*}{\partial t} + \frac{U}{\Omega L}\vec{q}^*.\nabla\vec{q}^* + 2\vec{k}\times\vec{q}^* =$$

$$= -\frac{\nabla p}{\rho\Omega U L} + \frac{\nu}{\Omega L^2}\nabla^2\vec{q}^*, \qquad (3)$$

$$\nabla.\vec{q}^* = 0. \qquad (4)$$

In contrast to the non-rotating case where the Reynolds number governs the flow, rotating fluid flows involve two parameters: the Ekman number $E = \frac{\nu}{\Omega L^2}$ † and the Rossby number $R_o = \frac{U}{\Omega L}$. They indicate the ratios of respectively viscous to coriolis forces and inertial to coriolis forces. A general inspection of the basic equations and a comparison of the magnitudes of various terms can reveal a great deal about the characteristic features of rotating fluid flows.

When the Ekman number is small we have a typical situation in which highest order derivatives multiply a small parameter. Consequently there arise thin frictional boundary layers along the bounding surfaces based on a balance of coriolis and

† In our thesis we have found it convenient to use the reciprocal of the Ekman number as the appropriate parameter instead of E .

viscous forces. These boundary layers were first noticed by Ekman [2] in his study of wind stress on the ocean surface, and are found to play an important role in the spin-up process in rotating fluids [3,4,5]. An excellent review of the spin-up problem is given by Benton and Clark [6]. A good deal of literature on rotating fluids is devoted to the study of various variants of the problem considered by Ekman [7,8,9,10].

Another type of boundary layers that arise in rotating fluids are associated with small Rossby number based on a balance of the coriolis and the convective accelerations [11,12]. These are commonly referred to as inertial boundary layers. At any solid boundary these inertial layers will themselves be accompanied by a narrower frictional layer [13,14].

When both R_o and E are small the dynamic balance, in steady flows, must be between the coriolis acceleration and the pressure gradient; in such motions the pressure variation will clearly be of order $\rho \Omega \omega L$. For unsteady flows the acceleration $\frac{dq}{dt}$ will contribute to this balance provided t_o is of order $\frac{1}{\Omega}$ or smaller. Such flows are called geostrophic flows.

Geostrophic flows display some remarkable features which are not encountered in non-rotating fluids. Taylor-Proudman theorem is perhaps the most striking feature of geostrophic flows. Another property of such flows becomes evident from the momentum equation for steady flows viz.

$$2\vec{k} \times \vec{q} = -\frac{1}{\rho} \nabla p. \tag{5}$$

From this it follows that the pressure gradient is perpendicular to the flow direction and hence the pressure is constant along a stream line (In non-rotating systems it is usual to think of pressure variations along a stream line.) This feature of geostrophic flows is particularly useful in weather forecasting [15,16].

Rotating fluids have an intrinsic stability in the sense that if a fluid particle is displaced from its equilibrium position of rigid body rotation, the coriolis force acts as a restoring force. If a fluid particle is displaced from its equilibrium radius, it will oscillate with twice the angular velocity of rotation about its equilibrium position. This frequency of oscillation is called the inertial frequency.

A rotating fluid subjected to an oscillatory disturbance will give rise to interesting physical phenomena. Owing to the oscillations, Stokes - Ekman layers will be formed along the bounding surfaces. These layers can be viewed either as classical Ekman layers modified by the imposed oscillations or as Stokes layers modified by rotation. If the forced frequency is equal to the inertial frequency the thickness of one of the layers becomes infinitely large and this presents difficulties in finding solutions for flow in semi-infinite expanse of fluids, satisfying all the boundary conditions [17,18,19, 20]. This difficulty can however, be overcome by considering finite domains [21].

Looking back one can say that most of the features of rotating fluids arise as a result of delicate balance and interplay of coriolis force, pressure gradient and viscous diffusion. The two key parameters are the Rossby number and the Ekman number. Another parameter which appears in the study of oscillatory flows is the frequency parameter $\alpha = \frac{\omega L^2}{\nu}$ where ω is the frequency of oscillation. The mechanics of any phenomena in rotating fluids will depend upon which parameters control the flow.

3. Literature survey and summary of the present work

The present thesis is divided into two parts. Part-A of the thesis is devoted to the study of oscillating source flow between rotating disks. Part-B deals with the study of flow in rotating pipes of arbitrarily varying gap. This section is devoted to a brief review of the literature directly related to problems studied in this thesis.

(a) Source Flow

Laminar source flow between two parallel disks has been studied theoretically and experimentally by a number of workers in view of its application in the design of heat exchangers, gaseous diffusion, lubrication of bearings, leakage flow over the shroud of centrifugal pump or compres-ssor, acoustics and biomedical engineering.

The study of this problem may be sub-divided into two groups:

(1) Source flow between stationary disks.

(2) Source flow between rotating disks.

Source flow between parallel stationary disks has been investigated by Livesey [22], Peube [23], Savage [24], Sourieau [25] and Moller [26].

Livesey used the momentum integral method with a parabolic velocity profile while Peube, Savage and Sourieau used an expansion procedure valid for large values of r , the radial co-ordinate. The expression for the radial velocity component turns out to be a series in ascending powers of $\frac{R_e}{r^2}$ where $R_e = \frac{m}{\nu h}$ is the Reynolds number based on the flow rate m . Their theoretical results are in better agreement with the experimental data of Moller than those of Livesey.

Source flow between two parallel disks rotating with the same angular velocity has been considered by Pohlhausen and Breitner [27] and Kreith and Peube [28], while the source flow between two parallel disks rotating with different angular velocities has been studied by Kreith and Viviand [29]. The method of solution in these studies is basically the same viz. expansion in powers of $\frac{R_e}{r^2}$

Surprisingly all these authors were unaware of the work of Soo [30] who used an entirely different approach. He considered the case when one disk is rotating with angular velocity ω while the other is at rest. By assuming the strength of the source m to be small such that squares and higher powers can be neglected the flow field is assumed as

$$u = -R_e r \omega W'(\zeta) - \frac{m U'(\zeta)}{2\pi \rho z_o r} \tag{6}$$

$$w = 2 R_e \omega z_o W(\zeta) \tag{7}$$

$$v = r \omega V(\zeta) \tag{8}$$

where $R_e = \frac{\omega z_o^2}{\nu}$, $\zeta = z/z_o$ and z_o is the distance between the plates. The resulting ordinary nonlinear differential equations for U , V and W are integrated numerically.

In all these investigations it was assumed that the strength of the source is constant. A different class of problems arise when the flow rate oscillates harmonically in time. The only available work published on oscillatory radial flow is due to Na, Nielsen and Grossman [31] and Elkouh [32] for flow between stationary disks. Na et al. considered the case when the flow rate has a small oscillatory component superimposed on a finite non-zero mean value. Their analysis is, however, restricted to small frequencies only. Elkouh on the other hand considered the case when the flow varies harmonically about a zero-mean value. The flow functions are expanded in terms of Reynolds number based on the amplitude a_0 of the source. The solution is valid for all frequencies.

In the present thesis we have studied the problem of oscillatory source flow between two disks rotating with angular velocity $\bar{\Omega}$ about an axis perpendicular to their plane. In the absence of the source the fluid and the disks will be in a state of rigid body rotation. The oscillatory radial diffusion gives rise to a complicated flow field which displays several interesting features. In particular the interaction of the coriolis, viscous and inertia forces gives rise to modified Stokes and Ekman layers adjacent to the two plates.

The problem in non-dimensional form involves three parameters: the Reynolds number, $R = a_0 / 4\pi\nu a$, the rotation parameter, $K^2 = \bar{\Omega} a^2 / \nu$ and the frequency parameter $\alpha = \bar{\omega} a^2 / \nu$, where a_0 is the amplitude and $\bar{\omega}$ the frequency of oscillation of the source, ν the kinematic viscosity and $2a$ is the distance between the disks. The solution is obtained by assuming R to be small and expanding the flow functions in powers of R. The first order harmonic and second order steady mean velocity is obtained and the effect of the parameters K^2 and α on the flow characteristics is studied.

Asymptotic solutions when K^2 or α is large are derived. For large K^2 the steady mean flow near the disks has a three layered structure: one of which is pure Ekman layer of thickness of $O\left(\frac{1}{K}\right)$ and the other two may be identified as modified Ekman layers of thickness of

$$O\left\{\frac{1}{K}\left(1 - \frac{\alpha}{4K^2}\right)\right\} \quad \text{and} \quad O\left\{\frac{1}{K}\left(1 + \frac{\alpha}{4K^2}\right)\right\} .$$

On the other hand for large α the mean flow near the disks has a two layered structure. These layers may be referred to as the modified shear layers of thickness of

$$O\left\{\sqrt{\frac{2}{\alpha}}\left(1 + \frac{K^2}{\alpha}\right)\right\} \quad \text{and} \quad O\left\{\sqrt{\frac{2}{\alpha}}\left(1 - \frac{K^2}{\alpha}\right)\right\}.$$

(b) Flows in rotating ducts

When a viscous incompressible fluid flows in a channel rotating about an axis perpendicular to the axis of the channel, the coriolis force gives rise to a secondary flow in a direction perpendicular to both the axis of rotation and the primary flow. (The primary flow may be due to a prescribed pressure gradient or flow rate). As a result of the triangular interaction of the coriolis force, pressure gradient and viscous diffusion the flow field displays many interesting features. In particular for small Ekman number there arise thin boundary layers near the boundaries which partly control and are partly controlled by the geostrophic core.

Barua [33] was the first to show that, when a straight pipe of circular cross-section through which liquid is flowing under a constant pressure gradient rotates with a constant angular velocity Ω about an axis perpendicular to its own, secondary motion is set up and the fluid particles move in spirals relative to the pipe. The solution is obtained by expanding the flow functions in a series of ascending powers of the Taylor number $2a^2\Omega/\nu$ and is valid only for small values of the speed of rotation. It is found that (i) secondary flow is qualitatively similar to that observed in a stationary curved pipe [34,35] (ii) pressure gradient remaining the same rotation tends to decrease the flux through the pipe.

Benton [36], unaware of the work of Barua, considered the same problem and arrived at similar conclusions. In a subsequent paper Benton and Boyer [37] considered the more interesting problem of flow in a rapidly rotating pipe of arbitrary cross-section. Their main results are:

(i) there arise well developed thin boundary layers of the Ekman type along the channel walls in which viscous effects are important.

(ii) the flow in the interior is geostrophic, and

(iii) inertia effects are negligible in the whole of fluid region.

Benton and Boyer however did not consider the shear layers parallel to the axis of rotation. Recently Hocking [38] has carried out a detailed analysis of the structure of shear layers on boundaries parallel to the axis of rotation. The specific problem considered by him is the flow in a pipe of rectangular cross-section with its axis curved in a circle and rotating with angular velocity Ω about an axis normal to the plane of the circle. His main results are:

(i) There are two shear layers of thickness $O(E^{1/3})$ and $O(E^{1/4})$.

(ii) The function of these layers is to reduce the axial flow to rest at the wall, to stop the outward flow and turn it into a vertical flow (parallel to the axis of rotation) and finally to reduce this vertical flow to rest at the wall.

In the terminology of Hocking these layers may be referred to as Stewartson - Proudman layers [39,40].

A relatively simple problem was considered by Vidyanidhi and Nigam [41]. They analysed the flow under a constant pressure gradient in a rotating channel formed of two infinite parallel plates. The importance of their solution lies in the fact that it constitutes an exact solution of the Navier-Stokes equations. Vidyanidhi, Bala Prasad and Ramana Rao [42] have recently extended it to include suction and injection at the plates.

In all these studies mentioned above the flow is assumed to be steady and fully developed. The study of oscillatory flows in rotating ducts has not received attention. As mentioned earlier, oscillatory flows in rotating fluid can give rise to interesting physical phenomena.

In the present thesis we have initiated the study of oscillatory flows in rotating ducts. The specific problem considered is the rotating flow in a symmetric duct of varying gap when the flow rate oscillates harmonically in time with frequency ω . We assume that the channel gap varies slowly with the longitudinal coordinate x . The equations of motion are simplified by using standard boundary layer approximations. The solution is obtained by a process of successive approximations. The first-order harmonic and the second-order steady mean velocity are obtained. Numerical results are given for a diverging channel given by

$$y = \pm \frac{a}{2}\left(1 + e^{\frac{\epsilon x}{x_0}}\right)$$

The interaction of the imposed oscillations and coriolis force gives rise to Stokes-Ekman layers near the walls.

The structure of these layers is examined in details. The solutions for the following problems are deduced as particular cases of the general results:

(i) Oscillatory flow in a non-rotating duct of varying gap.

(ii) Oscillatory flow in a rotating duct of constant gap-width.

(iii) Steady flow in a rotating duct of varying gap.

In the last chapter we have considered the problem of steady flow in an asymmetric duct of varying gap. The walls of the duct are given by

$$y_1 = a\left[1 + \epsilon_1 g_1(x)\right]$$

and

$$y_2 = -a\left[1 + \epsilon_2 g_2(x)\right]$$

where x and y are respectively the co-ordinates along and perpendicular to the axis of the duct. ϵ_1 and ϵ_2 are the rate of divergence parameters and $g_1(x)$ and $g_2(x)$ are the upper and lower wall variations relative to the mean half width of the channel denoted by a. In addition to ϵ_1 and ϵ_2 the problem in its non-dimensional form involves three other parameters: R_o the Rossby number, K^2 the rotation parameter and δ a measure of the slope of the channel. The effect of the wall deformations on the secondary flow is studied. It is found that in a rapidly rotating system the core velocity depends on the gradients of the wall deformation. For the purpose of illustration numerical results are given for the following particular cases:

(i) Channel with sinusoidal wall deformations out of phase (Symmetric channel).

(ii) Channel with sinusoidal wall deformations in phase (Asymmetric channel).

(iii) Channel with sinusoidal wall deformations having a phase difference ϕ (Asymmetric channel).