INTRODUCTION AND SUMMARY

The dynamic behaviour of realistic control systems is described by ordinary differential equations of high order. However, in a majority of such systems, the behaviour is governed by a few dominant parameters, a relatively minor role being played by the remaining small parameters, such as time-constants, masses, moments of inertia, inductances and capacitances.

Such systems are in general modelled by differential equations of the form,

$$\frac{dX}{dt} = F(X,Z,U,t,h), \quad X(t=0) = X(0), \quad \text{specified} \qquad \ldots (A)$$

$$h\,\frac{dZ}{dt} = G(X,Z,U,t,h), \quad Z(t=0) = Z(0), \quad \text{specified} \qquad \ldots (B)$$

where X and Z are respectively n- and m-dimensional state vectors, U is an r-dimensional control vector, and <u>h</u> is a small positive parameter. Referring to equations (A) and (B), it is readily seen that letting $h = 0$ results in an n-dimensional lower order system. Such equations can be conveniently tackled using a perturbation technique referred to as the singular perturbation technique[28] (SPT). From an engineering point of view, the SPT is equivalent to obtaining approximate solutions of a higher order system by solving a lower order model.[12] As such it is very attractive to formulate the real problems in control into the framework of the mathematical theory of singular perturbations which has an extensive literature.[3,11,14,16,28]

The singular perturbation technique, originally developed for the analysis of high order dimensional initial value problems, described by linear and nonlinear equations,[3,26-28] has recently been successfully extended to several types of problems arising in optimal control theory.[9,12,13,16,23,29,30] The technique is finding its way into areas like time-delay systems,[25] distributed parameter systems,[2] and estimation theory.[8,19] Its engineering applications embrace such wide disciplines as electrical power systems,[21] fluid mechanics,[10] flight mechanics,[1] soil mechanics,[6] nuclear reactor dynamics,[2,20] geophysics,[5] and biology.[22]

Some of the earliest contributions of significance to singular perturbation theory are the works of Tikhonov[26] and Vasil'eva.[27] Vasil'eva obtained the solution of initial value problems in the form[28]

$$X(t,h) = \sum_{j=o}^{\infty} X^{(j)}(t)h^j + \sum_{j=o}^{\infty} \bar{X}^{(j)}(t')h^j - \sum_{j=o}^{\infty} \underline{X}^{(j)}(t')h^j$$

$$Z(t,h) = \sum_{j=o}^{\infty} Z^{(j)}(t)h^j + \sum_{j=o}^{\infty} \bar{Z}^{(j)}(t')h^j - \sum_{j=o}^{\infty} \underline{Z}^{(j)}(t')h^j \qquad \ldots (C)$$

where $t'h = t$ is the 'stretched' transformation. The above three series are called the outer, inner, and intermediate series respectively. O'Malley[15] had developed a Boundary Layer Method for initial value problems. Subsequent developments in this field were mainly directed towards the application of SPT to optimal control problems and other areas.[7,12,16-18,24,30]

From a critical review of these works and a survey of literature[11] the following points emerge :

(i) no attempt has been made so far to solve engineering problems using Vasil'eva Method; and

(ii) all works relate to Boundary Layer Method and, furthermore, solutions have not been obtained for approximations higher than the first order in all the problems considered so far. This should be attributed to the fact that the formulation as well as obtaining solutions of equations using Boundary Layer Method is quite unwieldy for higher order approximations.

The main object of the present investigation is to present a unified approach to control problems (amenable for singular perturbation analysis) making use of Vasil'eva Method on lines originally suggested by Vasil'eva for initial value problems. It is shown that one of the main advantages of this method is its simplicity in the formulation of equations for solving the inner- and imtermediate-series functions. In fact, the formulation of equations for inner-series functions, upto any degree of approximation, can be made by mere inspection of the outer-series equations. Furthermore, the nature of intermediate-functions is such that it is a polynomial in t', and can be written down from inspection. In the present investigation the solutions have been obtained upto second order approximation for most of the problems considered.

The application of SPT to two-point boundary value problems (TPBVP) arising in optimal control has been made by several authors.[7,9,11,16,18,23,29] An examination of their solutions reveals that their approach does not meet all the boundary conditions exactly, although, the errors involved for the problems considered are small. In this investigation a modification of the Vasil'eva condition for the determination of boundary conditions for the outer functions is proposed which meets all boundary conditions exactly.

Vasil'eva Method has been applied and the results critically examined for the following types of control problems :

(1) the linear and non-linear initial value problems with or without inputs (Chapter 1) ;

(2) the terminal (final) value problems involving the normal and inverse matrix Riccati equations in the closed-loop optimal control (Chapter 2) ; and

(3) the TPBVP arising in the open-loop optimal control (Chapter 3).

In each of the above cases the effect of the variation of the small parameter $\underline{h}$, and the boundary conditions is examined. In addition, the relative accuracy of the solutions of the zeroth, first, and second order approximations is studied by considering the integra-squared-error of the above approximate solutions with respect to the high order solution.

So far the extension of the SPT to solve the analogous problems associated with high order discrete systems has not been attempted in the literature,[11] presumably due to the apparent difficulty in visualizing for such systems the analog of the 'stretched time' functions associated with the continuous systems. In Chapter 4, a beginning is made to extend the SPT to discrete systems which are the discrete models of the high order continuous systems.[4] In this connection, the 'stretched time equivalent' for discrete models is first obtained by establishing a one-to-one correspondence between the continuous system and its discrete model. Based on this equivalence, the SPT is extended to discrete systems as well. This technique is applied to initial value problems, boundary and final value problems arising in optimal control and is illustrated by several examples. Solutions are obtained upto fourth order approximation and results are compared with the corresponding solutions of continuous systems. A critical examination of solutions of discrete models and continuous systems reveals certain interesting similarities.