CHAPTER 0

GENERAL INTRODUCTION

The basic unsolved problem in the representation theory of algebraic Chevalley groups defined over a field of characteristic $p \neq 0$ and their Lie algebras is the computation of characters of the irreducible representations. In his seminal work Verma [24] reduced the problem to that of determining a finite set of integers $C_{\lambda\mu}$. He went very deep into the structure of affine Weyl groups and came up with results about representation theory which involved Harish-Chandra Principle, Humphrey's numbers etc. He also made few conjectures which depended upon the properties of the affine Weyl groups. All his conjectures were proved to be true. One such conjecture was about the Weyl's dimension polynomial. Given a complex semisimple Lie algebra g_c of rank n, there is a homogeneous polynomial D known as Weyl's dimension polynomial, in n variables with rational coefficients, which yields the dimension of the irreducible g_c-module when the variables are given positive integral values. He introduced certain affine transformations w_σ for σ belonging to Weyl group W of g_c. Denoting by D_σ the polynomial obtained by transforming D under w_σ he conjectured that $\sum b_\sigma D_\sigma = 1$, for unique integers b_σ. This conjecture was proved in the affirmative in [9]. The proof involved a certain matrix which imposes a new partial order on Weyl groups. This matrix has been further investigated and used to find some results on representations of algebraic Chevalley groups [3] . The same matrix is the weighted incidence matrix for our definition of graphs on Weyl groups.

The aim of this thesis is to study the graph structure on Weyl groups. We study some basic properties of such graphs. First we prove some properties which are peculiar to the graphs on Weyl groups. Then we prove results about graph automorphism connectivity, planarity and girth. We hope that our investigations on the graphs on Weyl groups will pave a way to interesting results on the representation theory of algebraic Chevalley groups and their Lie algebras.

The chapterwise arrangement of this thesis is as follows :

In Chapter I, first we give some preliminaries on the root systems, Weyl groups and Dynkin diagrams etc. Then we give results on Affine Weyl groups and discuss the Verma's conjecture on Weyl's dimension polynomial. Next we give the details of the proof of Verma's conjecture [9] . The results given there suggest the definition of the graphs on Weyl groups. Finally, we describe some concepts and results from graph theory which are required by us later.

Chapter II deals with the basic structure of our study, namely the graphs on Weyl groups. The vertices of such a graph are elements of a Weyl group W and the edges are defined through a relation $\longrightarrow$ on the Weyl group W which involves it associated root system Φ. The resulting graph is denoted by $\Gamma(W(\Phi))$ which we also write as $\Gamma(W)$ or $\Gamma(\Phi)$. First we prove some necessary and sufficient conditions for $\sigma \longrightarrow \tau$ when σ, τ are elements of a Weyl group W. Next we show that if W_J is a certain Weyl subgroup of W and $\sigma, \tau \in W_J$ then (σ, τ) is an edge in $\Gamma(W_J)$ iff (σ, τ) is an edge in $\Gamma(W)$. This gives a very fundamental result that $\Gamma(W_J)$ is an induced subgraph of $\Gamma(W)$. This result on induced subgraph is used repeatedly throughout this thesis. Next we obtain some graph automorphisms of $\Gamma(W)$. These graph automorphisms arise due to multiplication by elements of a certain subgroup Ω_0 of W. To derive this we study the action of the affine Weyl groups on the interior points of the fundamental simplex. We also show that $(\sigma, \gamma\sigma)$ is never an edge in $\Gamma(W)$ for $\sigma \in W$ and $\gamma \in \Omega_0$.

In Chapter III, we study the connectivity of graphs on Weyl groups. First we prove that the graph on the Weyl group of type A_n for $n \geqslant 4$ is connected. The proof needs some very deep results on the properties of certain subgroup Ω_0 of W and the resulting automorphisms of the graph. We offer a conjecture on the connectivity of graphs on Weyl groups whose associated root systems are irreducible. We verify it for Weyl groups of type B_3, B_4, C_3, C_4, D_4 and G_2. The verificati is done by applying fusion method [4] to the claws of the graphs of the above mentioned groups. We also study the structure of graphs on direct product of Weyl groups. We show that if W_1 and W_2 are Weyl groups then $\Gamma(W_1)$ and $\Gamma(W_2)$ are connected iff $\Gamma(W_1 \times W_2)$ is connected. From this we deduce that $\Gamma(W)$ is connected if the associated root system of W does not have the root system of type A_1, A_2, A_3 or B_2 as a factor.

Chapter IV deals with the planarity of graphs on Weyl groups. First we study the case when the root system involved is irreducible. We show that $\Gamma(\Phi)$ for irreducible root system Φ is nonplanar except when Φ is of type A_1, A_2, A_3, B_2 and G_2. This is proved by showing that the graphs for Weyl groups of type A_4, B_3, C_3 and D_4 are nonplanar and then invoking the theorem on induced subgraph. The case of arbitrary Φ is determined after establishing few technical results about graph on direct product of Weyl groups. One of the crucial result we prove is that if a Weyl group W is equal to $W_1 \times W_2 \times W_3$ such that each of $\Gamma(W_1)$, $\Gamma(W_2)$ and $\Gamma(W_3)$ has at least one edge then $\Gamma(W)$ is nonplanar. This reduces the whole problem of planarity essentially to that of a graph on direct product of two irreducible root systems. We prove that $\Gamma(\Phi)$ is planar iff Φ is equal to Φ_1 or Φ_2 or else $\Phi_1 \times \Phi_2$ where Φ_1 is a root system of type $A_1^{k_1} \times A_2^{k_2}$ with k_1, k_2 non-negative integers and Φ_2 is any one of the following root systems : A_3, B_2, G_2, $A_3 \times A_3$, $A_3 \times B_2$, $A_3 \times G_2$, $B_2 \times B_2$ and $B_2 \times G_2$.

In Chapter V , we determine the girth of the graphs on Weyl groups. Let $g(\Gamma(\Phi))$ denote the girth of the graph $\Gamma(\Phi)$ for a root system Φ . First we show that (i) $g(\Gamma(\Phi))$ is 3 when Φ is any one of the following types : A_7, B_3, C_3, D_4 and (ii) $g(\Gamma(A_4))$ is 4 . Next we show that if Φ is irreducible root system then (i) $g(\Gamma(\Phi))$ does not exist if Φ is of type A_1, A_2, A_3 or B_2, (ii) $g(\Gamma(\Phi))$ is 4 if Φ is of type A_4 or G_2, (iii) $3 \leq g(\Gamma(A_5))$, $g(\Gamma(A_6)) \leq 4$ and (iv) in all the remaining cases $g(\Gamma(\Phi))$ is 3 . By using the properties of graphs on direct product of Weyl groups we show that the results of irreducible Φ can be generalised to arbitrary Φ with minor modifications. We prove that $g(\Gamma(\Phi))$ is 3 except in very few cases.

Finally, in Appendix, we include the important information about the low rank irreducible root systems, their Weyl groups and the relevant data which gives the graph structure studied in this thesis. The root systems included are of type A_1, A_2, A_3, A_4, B_2, B_3, B_4, C_3, C_4, D_4 and G_2 . The data includes σ , c_σ and $D(-c_{\sigma\sigma_0} + c_\tau)$ for the Weyl groups mentioned above. All the computations have been done on a computer. The Verma integers b_σ also have been listed here which requires extra computations. The method of computation employed is as described in [9] . These values of b_σ have a direct bearing on the representations of algebraic Chevalley groups and their Lie algebras over a field of characteristic $p \neq o$ [24] .