

SYNOPSIS

The thesis comprises the study of Rayleigh quotient, coupled Rayleigh quotient, numerical range and coupled numerical range of an operator and that of a new functional defined on a Hilbert space. It consists of six chapters and contain some lemmas, about forty theorems and a number of corollaries.

CHAPTER 1. The notion of two sequences of vectors in a Hilbert space H tending to linear dependence has been introduced. Using a property of these sequences it has been established that $\sup |(Ag, g)| = \|A\|$, $\|g\| = 1$ and $\{f_n\}$ be a weakly convergent sequence of unit vectors, with weak limit f , such that $|(Af_n, f_n)| \rightarrow \|A\|$ then $\{(Af_n, f_n)\}$ also converges and the limit is attained if $f \neq 0$, i.e., if the limit is not attained the weak limit must be zero.

Also, it has been proved that if A^p is normal and compact, $p > 1$ and $\|A^p\|_S = \|A\|_S^p$ for all subspaces S which reduce A , then A itself must be normal and compact.

An improvement of the well-known inequality $N_A \geq \|A\|/2$ has also been given, where $N_A = \sup |(Af, f)|$, $\|f\| = 1$.

In fact it has been proved that

We have used $w(A)$ in place of N_A in all the chapters.

$$N_A \geq \lim_{n \rightarrow \infty} \left[\frac{\|A\|}{2} \cdot \frac{\|A^2\|^{1/2}}{2^{1/2}} \cdots \frac{\|A^{2^n}\|^{1/2^n}}{2^{1/2^n}} \right]^{\frac{1}{n+1}} \geq r(A),$$

where $r(A)$ is the spectral radius of A .

By using a variational procedure, it has been established that at the extrema of Rayleigh quotient all operators evince a normal-operator-like behaviour.

Finally a necessary and sufficient condition that an operator is normal has been given as follows:

If A be such that $\sup_{f \in S} |(Af, f)| / (f, f)$,

where S is any reducing subspace of A , is always attained for an eigenvector of A belonging to S and that the non-zero eigenvalues have finite multiplicity with only possible limit point zero, then the operator is normal and compact.

CHAPTER 2. Here we establish that a necessary and sufficient condition for the Rayleigh quotient $R(f)$ to be stationary at a vector $f \in H$ is $\bar{\lambda} Af + \lambda A^* f = 2|\lambda|^2 f$,

$\lambda = (Af, f) / (f, f)$. A number of corollaries flow out from this theorem, an important one being that an operator behaves like a normal one at all stationary values of its Rayleigh quotient. The well known inequality $N_A \geq \|A\|/2$ has

been sharpened to $N_A \geq \|A\|/\sqrt{2}$ for operators for which $\sup |R(f)|$ and $\sup \|Af\|/\|f\|$ are attained for the same vector. An elementary approach to the spectral theory of compact normal operators has also been outlined.

It has also been proved that if a weakly convergent sequence $f_n \rightharpoonup f$, $\|f_n\| = 1$ be such that $\{(Af_n, f_n)\}$ converges and $|(Af_n, f_n)| \rightarrow N_A$, then the supremum is attained if $f \neq 0$. In other words, if the supremum is not attained, all such weakly convergent sequences must tend weakly to zero only.

Some orthogonality properties of stationary vectors have been discussed. It has also been shown that a stationary vector of a compact normal operator can be expressed as a linear combination of a finite number of its eigenvectors.

CHAPTER 3. We define a new quotient $R_B(f) = (BAf, f)/(Bf, f)$, hereafter called the coupled Rayleigh quotient of A for the auxiliary operator B , (or briefly coupled Rayleigh quotient). We do not assume B to be positive definite bounded below. The motivation is to characterize eigenvalues of non-normal operators variationally.

It has been shown that if the eigenvalues of a normal operator A can be obtained from $R_B(f)$ variationally, where B is self-adjoint then B reduces the eigenspaces of A corresponding to each eigenvalue and commutes with A .

Let B be self adjoint and invertible and let

$$AB^{-1}A^*B = B^{-1}A^*BA \quad \text{then the eigenvalues of } A \text{ can be}$$

obtained in the recursive variational manner from $R_B(f)$ if the restriction of B to the range of A is positive-definite. The ordinary Rayleigh quotient fails in this case to yield the eigenvalues variationally. The coupled numerical range of A has been defined as

$$W_B(A) = \left\{ z : z = \frac{(BAf, f)}{(Bf, f)}, (Bf, f) \neq 0 \right\}.$$

This set is not convex in general but it contains interior of line segments joining such points f and g that make $(Bf, f)/(Bg, g)$ positive. The closure of coupled numerical range contains the spectrum of A if A commutes with B . If the coupled numerical range is assumed to be convex for some B then the assumption $AB=BA$ can be dropped.

CHAPTER 4. In this chapter different Rayleigh quotients corresponding to the eigenvalue problem $Af = \lambda Bf$ have been studied. It has been shown that at stationary values of

the Rayleigh quotient $(Af, f)/(Bf, f)$, we have the eigenvalue relation $Af = \lambda Bf$, if A and B are self adjoint only, not necessarily positive definite.

Again, stationary values of $(Af, Bf)/(Bf, Bf)$ also lead to the eigenvalue relation $Af = \lambda Bf$, if A commutes with B^* . We establish further that the numerical range

$$W_B(A) = \left\{ z : z = \frac{(Af, f)}{(Bf, f)}, (Bf, f) \neq 0 \right\}$$

is not convex in general; but contains interior of line segments joining such points f and g that make $(Bf, f)/(Bg, g)$ positive, whereas the numerical range

$$W'_B(A) = \left\{ z : z = \frac{(Af, Bf)}{(Bf, Bf)}, Bf \neq 0 \right\}$$

is always convex.

CHAPTER 5. In this chapter it has been shown that an operator always has eigenvalues at the corners of its numerical range.

Also if $\lambda \in \partial W(A)$ and $f_n \rightarrow f$ be such that $(Af_n, f_n) \rightarrow \lambda$ then either $\lambda \in W(A)$ or $f = 0$.

A measure of how much of the exterior of the line segment joining $\xi = (Af, f)/(f, f)$ and $\eta = (Ag, g)/(g, g)$ also lies in it, has been obtained. Some geometrical properties of the vectors on the boundary of the numerical range have also been established.

CHAPTER 6. In this last chapter a new functional

$$T(f) = \|Af\|^2 - |(Af, f)|^2, \quad \|f\| = 1$$

has been defined with the motivation of finding eigenvalues of A from $T(f)$ variationally. The new concept of vectors at a stationary distance from being an eigenvector, in short S.D. vectors, has been introduced.

It has been shown that the functional $T(f)$ gives the deviation of a vector f from being an eigenvector of A . So, by variational method, we have characterized all vectors at a stationary distance from being an eigenvector, (in particular vectors nearest to or farthest from being an eigenvector).

The characterization is the following: The necessary and sufficient condition that f is a stationary distance vector is

$$(A^* - \bar{\lambda})(A - \lambda)f = \|h\|^2 f,$$



where $\lambda = (Af, f)$, $\|f\| = 1$ and $\|h\|^2 = T(f)$.

If A is self adjoint then S.D. vectors are of the form $c_i f_i + c_j f_j$, $|c_i|^2 = |c_j|^2 = 1/2$ where f_i, f_j are eigenvectors. We have indicated a method of obtaining the eigenvalues of a self adjoint compact operator by maximizing the functional $\|Af\|^2 - |(Af, f)|^2$. In fact the method works also for operators with complete eigenvalues even if an eigenvalue is of infinite multiplicity. The usual Rayleigh quotient method, however, does not seem to work.

The S.D. vectors of normal operators are also shown to have similar properties.