A GEOMETRIC PROSPECTIVE OF DIRECTIONAL DATA ANALYSIS

Abstract

Directional data analysis is commonly based on trigonometric representations, with limited exploitation of the intrinsic geometry of its sample space. Although the geometry is naturally incorporated for circular and spherical data, bivariate angular data are typically modelled on a flat torus, thereby ignoring curvature-induced dependency. This thesis develops a geometric framework for directional data analysis, in which statistical modelling, computation, and inference are formulated intrinsically on circular, toroidal, and spherical manifolds using their area elements.

The thesis first addresses computational challenges by developing efficient sampling methods for directional distributions on the circle, sphere, and torus. A modified acceptance–rejection algorithm, based on piecewise-constant envelopes derived from upper Riemann sums, is proposed. It provides higher acceptance rates and reduced computational cost. Within this framework, a geometric analog of the von Mises distribution on a curved torus is introduced, capturing directional concentrations and curvature-induced dependency.

Exploiting the area element of the curved torus, a flexible five-parameter dependent distribution is derived from the area-uniform measure. The model admits tractable marginals and a conditional distribution belonging to the cardioid family. This enables circular–circular regression via conditional expectations. An exact sampling scheme for cardioid distributions is also developed. The methodology is demonstrated through postoperative astigmatism data.

The area element motivates a novel angular loss function, termed the “square of an angle (SoA),” which serves as an analogue of least squares for directional data. This loss leads to a semi-parametric linear–circular regression model based on generalised Möbius transformations, requiring no distributional assumptions on angular errors. The model is illustrated using cryptocurrency data.

The concept of SoA is further extended to define a curved dispersion matrix and a Mahalanobis-type distance on a torus and a sphere. This yields distribution-free tests for change-point detection in toroidal and spherical mean directions. The test statistics weakly converge to the Kolmogorov distribution under the null hypothesis and are consistent under location alternatives. Applications to wind–wave directional data and the trajectory of the cyclonic storm *Biporjoy* demonstrate the methodology.

Keywords: Directional data, Upper Riemann sums; First fundamental form; Area element; Möbius transformation; Change-point detection