Abstract

In algebraic graph theory, researchers have considered several graph-associated polynomials to study the determination of graphs. The main objective of this thesis is to investigate the determination of some graphs by permanental polynomials of their Laplacian and signless Laplacian matrices. For a simple undirected graph G on n vertices, its Laplacian and signless Laplacian matrices are defined by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively, where $D(G)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix of G. The permanent of a $k \times k$ matrix M with entries m_{ij}, $i, j = 1, 2, \ldots, k$, is defined as $\mathrm{per}(M) = \sum_{\sigma} \prod_{i=1}^{k} m_{i\sigma(i)}$, where the sum is taken over all permutations σ of $\{1, 2, \ldots, k\}$. The permanental polynomial of M is defined as $\psi(M; x) = \mathrm{per}(xI_k - M)$, where I_k is the identity matrix of order k. Polynomials $\psi(L(G); x)$ and $\psi(Q(G); x)$ are known as Laplacian and signless Laplacian permanental polynomials of G, respectively. Many structural properties of a graph can be extracted from these polynomials. For instance, the multiplicity of root 1 of $\psi(Q(G); x)$ is equal to the star degree of G.

A graph G is said to be determined by its Laplacian (signless Laplacian) permanental polynomial if all graphs having the same Laplacian (signless Laplacian) permanental polynomial as that of G are isomorphic to G. In this thesis, we compute the Laplacian and signless Laplacian permanental polynomials of some classes of graphs, namely, star, wheel, friendship graphs, and a class of caterpillar graphs $S_n^{(r)}$. We show that all star graphs, caterpillar graphs $S_n^{(r)}$ (for $r = 2, 3, 4, 5$), friendship graphs (up to 7 vertices), and wheel graphs (up to 7 vertices) are determined by their respective Laplacian (signless Laplacian) permanental polynomials. Moreover, we prove that friendship and wheel graphs, in general, are combinedly determined by their Laplacian and signless Laplacian permanental polynomials.

We characterize triangle-free graphs with two types of vertex degrees $d(> 1)$ and 1, in terms of their signless Laplacian permanental polynomials. The unicyclic graphs discussed here are $UC(r, d)$, where the unique cycle is C_r with all whose vertices have degree $d(> 2)$, and the remaining vertices not lying in C_r are pendant. We show that $UC(r, d)$ is determined by its signless Laplacian permanental polynomial for any $d(> 2)$ and $r = 3, 4, 5, 6$. We also prove that the unicyclic graph obtained from $UC(r, d)$ by making $d - 1$ new vertices adjacent to a pendant vertex is determined by its signless Laplacian permanental polynomial, for $r = 4, 5$. Further, we show that among all connected graphs $UC(r, 3)$, $r \geq 3$, is determined by its signless Laplacian permanental polynomial.

A balanced tree $T(h, d)$ is a rooted tree of height h, in which all the non-pendant vertices have degree $d(> 2)$ and all leaves have the same depth. We prove that $T(h, d)$ has the maximal star degree among all possible graphs with the same degree sequence. Moreover, we show that $T(2, d)$ is determined by its Laplacian and signless Laplacian permanental polynomials.

For a signed graph $\hat{G} = (G, \gamma)$, its Laplacian matrix is defined as $\mathcal{L}(\hat{G}) = D(G) - \mathcal{A}(\hat{G})$, where $D(G)$ is the degree diagonal matrix of G and $\mathcal{A}(\hat{G})$ is the adjacency matrix of $\hat{G}$. In this thesis, we introduce the Laplacian permanental polynomial of a signed graph $\hat{G}$ defined by $\psi(\mathcal{L}(\hat{G}); x) = \mathrm{per}(xI_n - \mathcal{L}(\hat{G}))$. We develop some recursive formulae for $\psi(\mathcal{L}(\hat{G}); x)$. Further, we study the polynomial $\psi(L(\hat{G}); x)$, when $\hat{G}$ is a balanced signed graph, a signed tree, a balanced signed bipartite graph, and a signed graph with only odd-length negative cycles. Moreover, we compute the general expressions for the first five coefficients and the constant term of $\psi(\mathcal{L}(\hat{G}); x)$. We also study the determination of some signed cycles by their Laplacian permanental polynomials and present a few examples of Laplacian co-permanental pairs of signed graphs.

Keywords: Permanent of a matrix, Laplacian permanental polynomial, signless Laplacian permanental polynomial, graph isomorphism, permanental determination, degree sequence, co-permanental graphs, bipartite graph, tree, star graph, wheel graph, caterpillar graph, friendship graph, unicyclic graph, pendant star, star degree, balanced tree, rooted tree, signed graph, recursive formulae, coefficients, signed cycles.