Abstract

In this thesis, we use the framework of the Erlangen programme to investigate geometries on two-dimensional homogeneous space under the action of the Lie group $\mathrm{SL}(3,\mathbb{R})$. To begin, we define the homogeneous space $\mathrm{SL}(3,\mathbb{R})/H$ as the real projective plane $\mathbb{RP}^2$, incorporating zero divisors through the action of $\mathrm{SL}(3,\mathbb{R})$ as projective transformations. We then analyse one-parameter subgroups (up to conjugacy) and their corresponding actions, resulting in natural invariants such as power curves, parabolas, and logarithmic spirals.

Next, we explore projective properties, including plane duality and cross ratio, for the $\mathrm{SL}(3,\mathbb{R})$ action. Additionally, we demonstrate that the map i associated with the $\mathrm{SL}(2,\mathbb{R})$ action cannot be extended for the $\mathrm{SL}(3,\mathbb{R})$ action. Subsequently, we focus on studying the projective action of subgroups on non-degenerate conics in $\mathbb{RP}^2$. Notably, the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ plays a significant role in one of the proofs, demonstrating that, subject to certain conditions, the fixed subgroup of the projective unit circle is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

Furthermore, we examine Möbius actions of $\mathrm{SL}(2,\mathbb{R})$, providing valuable insights into the associated elliptic, parabolic, and hyperbolic geometries. Consequently, we establish mappings from the elliptic, parabolic, and hyperbolic upper half planes to the interior, boundary, and exterior of the projective unit circle. These correspondences are expressed in a unified form as $f(u,v) = \left(\frac{u}{v}, \frac{1+\sigma v^2 - u^2}{2v}, \frac{1-\sigma v^2 + u^2}{2v}\right)$, where $u + \iota v = (u,v)$ and $\iota^2 = \sigma = -1,\ 0,\ 1$ for the elliptic (complex number system), parabolic (dual number system), and hyperbolic (double number system) cases, respectively. Within the context of these transformations, we note that cycles in the elliptic and hyperbolic cases are mapped to ellipses and hyperbolas, respectively. However, cycles are mapped onto the projective unit circle in the parabolic case due to degeneracy.

Finally, we extend the notion of cycle orthogonality. We initiate this exploration by examining the orthogonality of lines. Following that, we introduce rigorous characterizations of Möbius-invariant cycle images and establish the concept of orthogonality of quadrics within the projective disk.

Keywords: $\mathrm{SL}(3,\mathbb{R})$, Matrix Lie group, Erlangen program, projective transformation, homogeneous spaces, invariants, plane duality, cross ratio, Möbius-invariant cycles, conics, fixed points, derived representation, curvature, vector field, Iwasawa decomposition, real projective plane, adjoint representation, isotropy subgroup, EPH upper half plane, projective disk, cycle orthogonality.