CHAPTER I

INTRODUCTION

1.1 Preliminary Remarks

A 'Newtonian' fluid is characterised by a linear relation between stress and rate of strain of the form

$$\tau_{ij} = \mu e_{ij} , \qquad (1.1.1)$$

where τ_{ij} is the deviatoric stress tensor, e_{ij} is the strain-rate tensor and μ is the Newtonian viscosity. Its theory has been extensively investigated during the last century. It can successfully explain the phenomena of form drag, lift, skin friction, separation etc. However, it fails to explain the occurence of Weissenberg effect [1], Merrington effect [2] and Poynting effect [3]. These effects are particularly significant in materials like resins, pastes, plastics, high polymer solutions, certain varieties of oils and paints. In nature also materials like blood, honey, starch etc. exhibit these effects. The study of such materials has become essential due to their importance in modern industries. This has led to the formulation of various theories of non-Newtonian fluids. As a result of the inadequacy of the classical theory of Newtonian fluids many forms of constitutive equations have been proposed or are being proposed at present to explain the behaviour of such materials both theoretically and experimentally. In the following sections we shall give a brief review of the literature directly related to the present thesis which may help towards a better understanding of the results reported in the subsequent chapters.

1.2 Power-Law Fluids

These are the simplest class of non-Newtonian fluids in as much as the constitutive equation (1.1.1) is retained but it is assumed that μ is a function of strain-rate invariants. The constitutive equation, originally proposed by Ostwald [4], for incompressible power-law fluids may be expressed in the form

$$\tau_{ij} = K'^{n} \left| I_2 \right|^{\frac{n-1}{2}} e_{ij} , \tag{1.2.1}$$

where I_2 is the second invariant while K' and n are consistency and flow behaviour indices of the fluid respectively. It is to be noticed that the dimensions of K' depend on n . In a two-dimensional simple shearing flow of such fluids the apparent viscosity μ_a is given by

$$\mu_a = \frac{\tau_{xy}}{e_{xy}} = K'^{n} \left| \frac{du}{dy} \right|^{n-1} . \tag{1.2.2}$$

If the index n is less than one, the apparent viscosity decreases with the increase of rate of shear while the reverse holds for n greater than one. The former category of fluids will be referred to as pseudo-plastic while the latter will be called dilatant. Evidently $n=1$ corresponds to ordinary viscous fluids.

Theoretical studies of flows of power-law fluids have been carried out by a number of workers. As one might expect exact solutions are available only in few cases [5]. It is therefore necessary to obtain approximate solutions as in the case of Newtonian fluids. The most logical step in this direction would be to extend the concept of boundary layer to power-law fluids. Schowalter [6] obtained the boundary layer equations for power-law fluids in two-dimensions. Acrivos et al [7,8] studied the boundary layer flow past a semi-infinite flat plate while Kapur [9] considered the flow in a two-dimensional jet emerging from an orifice. Possible similarity solutions of the boundary layer equations in two-dimensions for power-law fluids have been obtained by Nanda [10]. Recently Hayasi [11] has reconsidered this problem and given some numerical results as well.

In all these investigations it is assumed that the flow is two-dimensional. We have in the present thesis initiated the study of boundary layer flows of power-law fluids in three dimensions. The boundary layer equations in stationary polar co-ordinates in the plane of the surface have been derived and possible similarity solutions of these equations have been obtained. As a particular case the problem of flow near an axially symmetric stagnation point has been completely solved. It is found that the shear stress at the plate is always greater than its value for ordinary viscous fluids.

Another field of interest has been the flow in pipes for chemical engineering applications. Bogue [12] was the first to discuss the flow of such fluids in the inlet length of a circular pipe. He has used the momentum integral method and obtained an approximate expression for the entry length required to establish the fully developed flow. The corresponding problem for flow in a straight channel has been discussed by Collins and Schowalter [13]. They have obtained two different solutions, one holding near the entry and the other holding near the fully developed region. They have also studied the flow in a circular pipe [14]. Recently Kapur et al. [15,16] have reconsidered in details the flow in the inlet length of a straight channel by using Bogue's method.

The study of heat transfer by laminar flow of power-law fluids in pipes and channels has not received much attention. Lyche and Bird [17] have extented the classical Graetz-Nusselt problem for flow in a circular pipe to power-law fluids, while Tien [18] has considered the heat transfer of power-law fluids flowing between two parallel plates. Other contributions in this field are by Schank and Van Laar [19], Hank and Christiansen [20] and Christiansen and Craig [21].

In all these investigations heating due to viscous dissipation has been neglected. In the present thesis we have considered the problem of heat transfer to power-law fluids for flow between two parallel plates when there may be heating due to viscous dissipation. The velocity field is assumed to be fully developed. The temperature field is determined both for the fully developed and the entrance regions. This enables us to find the inlet length required to reach the fully developed temperature field.

1.3 Second-Order Fluids

Merrington in 1943, observed some interesting results while measuring the rate of flow of rubber solutions or of oils containing metallic soap. He found that the liquid swelled on emerging from the tube. This anomalous effect could not be explained by the theory of classical viscous fluids. Merrington [2,22] proposed that this effect could well be due to elasticity of the fluids. This point of view was however not generally accepted [23,24,25]. A few years later Weissenberg [26,1] observed another interesting phenomenon . He found that when a rod is rotated in a cup containing a high polymer solution, the liquid rises up the stirrer. This phenomenon is referred to as 'Weissenberg effect' or 'normal stress effect'. Weissenberg [26,1,27] postulated that the fluids exhibiting these effects were visco-elastic in nature. Nevertheless, a satisfactory explanation could not be given at that time.

The first significant contribution towards the field of non-Newtonian fluids appeared in 1945 in a paper of Reiner [28]. In this paper he proposed a theory of non-linear viscosity and showed how such anomalous effects could be quantitatively accounted for by considering the second order terms. Reiner used the Stokes' definition of fluidity [29] that the stress is a function of strain rate. Assuming isotropy and homogeneity of the material, he expressed the constitutive equation as

$$\tau_{ij} = -p\,\delta_{ij} + \mu_1 e_{ij} + \mu_3 e_{i\alpha} e_{\alpha j}\,, \qquad (1.3.1)$$

where τ_{ij} is the stress and μ_1, μ_3 are functions of invariants. μ_1 has the dimensions of viscosity (M/LT) and can be reasonably called the coefficient of viscosity and μ_3 , though has the dimensions of M/L, is still called the coefficient of cross-viscosity. Using the constitutive equation (1.3.1) Rivlin [30] obtained exact solutions for shearing flow between two plates, flow in a cylindrical pipe and flow between rotating cylinders, while Srivastava [31,32] obtained exact solutions for flow due to a rotating disk and for flow near an axially symmetric stagnation point. Recently Jain [33,34] has reconsidered these problems using extremal point collocation method.

This theory has succeeded in explaining some of the phenomena such as 'Weissenberg effects', 'Merrington effects' and 'Poynting effects'. Experimental confirmation of theoretical predictions of this theory came from the works of Weissenberg [1], Garner and Nissan [35] and Lee and Warren [36]. However, some of the results based on this theory were later found not quite correct [37,38].

Further advance in the study of non-Newtonian fluids was made by Rivlin and Ericksen [39]. They postulated a visco-elastic fluid as isotropic and homogeneous in its state of rest, for which the stress components at any instant can be expressed as polynomials in the tensors formed by the gradients of velocity, acceleration and higher order accelerations. The coefficients of various terms that occur in the constitutive equation are further functions of the invariants of the tensors and the material properties. Further advance in this direction was made by Rivlin [40,41] who expressed the stress tensor in terms of eight material functions by assuming the dependence of stress tensor on gradients of velocity and first acceleration only.

Another significant development towards the study of mechanics of continuous media is due to Noll [42]. He has proposed a general theory of constitutive equations by assuming the dependence of stress on the history of deformation gradients only. Coleman and Noll [43,44,45, 46] have further developed these ideas by assuming certain approximation theorems for the functions involved in the constitutive equations. The constitutive equations for second-order fluids can be obtained as a particular case. According to this theory the constitutive equation in second order approximation has the following form

$$\tau_{ij} = -p\,\delta_{ij} + \mu_1 e_{ij} + \mu_2 B_{ij} + \mu_3 e_{i\alpha} e_{\alpha j}\,, \tag{1.3.2}$$

where B_{ij} is the second Rivlin-Ericksen tensor given by

$$B_{ij} = a_{i,j} + a_{j,i} + 2 v_{m,i} v_{m,j}\,, \tag{1.3.3}$$

and a_i is the acceleration vector.

A variety of hydrodynamical problems for non-Newtonian liquids using various constitutive equations have been solved by a number of workers during the last decade. Rajeshwari and Rathna [47] have considered the flow of second-order fluids near a stagnation point. In a subsequent paper [48] Rathna has discussed the flow due to a rotating disk, while Langlois [49,50] studied the steady-state visco-elastic flow and the flow between two rotating spheres. At the same time Langlois and Rivlin [51] discussed the slow steady-state flow of visco-elastic fluids through non-circular tubes. Recently Bhatnagar [52] has studied the secondary flow generated by the slow rotation of a sphere in second-order fluids. Srivastava [53,54] has discussed the flow of Reiner-Rivlin fluids and second-order fluids between two disks, one rotating and the other at rest. He has used a series expansion in terms of Reynolds number to solve the equations.

In the present thesis we have discussed the general problem of flow of second-order fluids past a symmetrical cylinder. The stream function for flow within the boundary layer is expanded in the form

$$\psi(x,y) = f_1(y)x + f_3(y)x^3 + f_5(y)x^5 + \cdots, \quad (1.3.4)$$

where y is the distance normal to the cylinder. The resulting ordinary differential equations for $f_1, f_3, \cdots$ have been integrated by Kármán-Pohlhausen method. As an example, the flow past a circular cylinder has been discussed for various values of the material constants characterising the fluid. It is found that the point of separation is nearer the front stagnation point than for Newtonian fluids.

Work on unsteady flows of non-Newtonian fluids has received attention only recently. The introduction of one more independent variable coupled with the fact that unsteadiness may enter a problem in a variety of ways makes the study of unsteady flows very complicated. Nevertheless some advance has been made in this direction. Markovitz and Coleman [55] have obtained exact solutions for the periodic helical flows of second-order fluids between two concentric cylinders or in a circular pipe while Pipkin [56] has given some general results for oscillating flow of non-Newtonian fluids in tubes of arbitrary cross-section. Srivastava [57] has discussed the flow of second-order fluids due to torsional oscillations of an infinite plate when the fluid is infinite in extent as well as the case when it is bounded by another stationary plate. A different class of unsteady problems are related to flows started impulsively from rest. Ting [58] initiated the study of such problems for second-order fluids. The specific problems treated by him are: generation of flows between two infinite parallel planes by constant tangential surface force, formation of flows through straight channels or pipes by constant pressure gradient and decay of parallel flows in channels or pipes when the pressure gradient suddenly ceases to act. In all these cases he has obtained exact solutions by Laplace transform technique. Schwarz [59] has discussed the Rayleigh problem of impulsive motion of an infinite flat plate by taking several different constitutive equations to represent the fluid. He has also considered the case when the plate performs harmonic oscillations in its own plane.

We have in the present thesis considered the more general problem of unsteady laminar boundary layer on a cylinder when the free stream oscillates harmonically in time about a zero mean. The general theory is applied specifically to the case of flow generated by a circular cylinder oscillating about a diameter in an infinite fluid.

1.4 Thermoviscous Fluids

The various theories of non-linear continuum mechanics described so far do not take into account the thermal cross-effects, although there is large amount of experimental evidence [60,61,62] which shows the dependence of the mechanical behaviour of various synthetic materials on temperature. Some work in this direction was done by Green and Adkins [63] and Truesdell [64]. A detailed and systematic attempt to incorporate thermal effects was made only recently by Koh and Eringen [65]. They have formulated a non-linear theory of thermo-viscoelasticity using the continuum mechanics approach. They postulate that a thermoviscous fluid in a thermal state is characterised by two sets of constitutive equations, one for stress tensor τ_{ij} and one for heat flux bi-vector H_{ij}, which are polynomial functions of the deformation rate tensor e_{ij} and the thermal gradient bi-vector b_{ij} with coefficients that are polynomials in the invariants of e_{ij} and b_{ij}. After some tedious algebraic calculations, the expressions for τ_{ij} and H_{ij} turn out to be

$$
\begin{aligned}
\tau_{ij} = {} & \alpha_1 \delta_{ij} + \alpha_3 e_{ij} + \alpha_5 e_{i\alpha} e_{\alpha j} + \alpha_6 b_{i\alpha} b_{\alpha j} \\
& + \alpha_8 (e_{i\alpha} b_{\alpha j} - b_{i\alpha} e_{\alpha j}) + \alpha_{12} (e_{i\alpha} b_{\alpha K} b_{Kj} + b_{i\alpha} b_{\alpha K} e_{Kj}) \\
& + \alpha_{15} (b_{i\alpha} e_{\alpha K} e_{Kj} - e_{i\alpha} e_{\alpha K} b_{Kj}) + \alpha_{17} (e_{i\alpha} e_{\alpha K} b_{Kl} b_{lj} \\
& + b_{i\alpha} b_{\alpha K} e_{Kl} e_{lj}) + \alpha_{20} (e_{i\alpha} b_{\alpha K} e_{Kl} e_{lj} - e_{i\alpha} e_{\alpha K} b_{Kl} e_{lj}) \\
& + \alpha_{22} (b_{i\alpha} e_{\alpha K} b_{Kl} b_{lj} - b_{i\alpha} b_{\alpha K} e_{Kl} b_{lj}) \\
& + \alpha_{24} (b_{i\alpha} e_{\alpha K} e_{Kl} b_{lm} b_{mj} - b_{i\alpha} b_{\alpha K} e_{Kl} e_{lm} b_{mj}),
\end{aligned}
\tag{1.4.1}
$$

and

$$
\begin{aligned}
H_{ij} = {} & \beta_1 b_{ij} + \beta_3 (b_{i\alpha} e_{\alpha j} + e_{i\alpha} b_{\alpha j}) + \beta_6 (e_{i\alpha} b_{\alpha K} b_{Kj} \\
& - b_{i\alpha} b_{\alpha K} e_{Kj}) + \beta_9 (b_{i\alpha} e_{\alpha K} e_{Kj} + e_{i\alpha} e_{\alpha K} b_{Kj}) \\
& + \beta_{12} (e_{i\alpha} e_{\alpha K} b_{Kl} b_{lj} - b_{i\alpha} b_{\alpha K} e_{Kl} e_{lj}) \\
& + \beta_{19} (e_{i\alpha} b_{\alpha K} b_{Kl} e_{lm} e_{mj} - e_{i\alpha} e_{\alpha K} b_{Kl} b_{lm} e_{mj}),
\end{aligned}
\tag{1.4.2}
$$

where $b_{ij} = e_{ijk}\,\theta_{,k}$ and $H_{ij} = e_{ijk}\,q_k$ with e_{ijk} representing the permutation symbol. The constitutive coefficients α_i and β_i are polynomials in the invariants of e_{ij} and b_{ij}, viz.

$$\begin{gathered} tr\, e_{ij}\,,\; tr\, e_{i\alpha} e_{\alpha j}\,,\; tr\, e_{i\alpha} e_{\alpha k} e_{kj}\,, \\ tr\, b_{i\alpha} b_{\alpha j}\,,\; tr\, e_{i\alpha} b_{\alpha k} b_{kj}\,,\; tr\, e_{i\alpha} e_{\alpha k} b_{k\ell} b_{\ell j}\,, \\ tr\, b_{i\alpha} e_{\alpha k} b_{k\ell} b_{\ell m} e_{mn} e_{nj}\,, \end{gathered} \tag{1.4.3}$$

with coefficients dependent on ρ (density) and θ (temperature).

The thermodynamical requirement of non-negative dissipation function gives

$$\alpha_3\, tr\, e_{i\alpha} e_{\alpha j} + \alpha_5\, tr\, e_{i\alpha} e_{\alpha k} e_{kj} \geqslant 0\,, \tag{1.4.4}$$

and $\beta_1\, tr\, b_{i\alpha} b_{\alpha j} \geqslant 0\,.$ (1.4.5)

However, since $tr\, b_{i\alpha} b_{\alpha j} = -2\left[b_{12}^2 + b_{23}^2 + b_{31}^2\right]$, we see that (1.4.5) implies $\beta_1 \leqslant 0$.

An essential feature of these constitutive equations is that it gives an expression for the stress due solely to the temperature gradient b_{ij} .

Koh and Eringen [66] analysed the flow of such fluids between two infinite parallel plates. Kelly [67] has considered some viscometric flows of such fluids. He finds that such simple flows do not exist for thermoviscous fluid in as much as the pressure is over determined by the equations of motion. From this he concludes that the formulation of the constitutive equations may be incorrect.

We have, in the present thesis, considered the more general problem of boundary layer flow of such fluids. We find that there is no such over determinacy as noticed by Kelly [67] and as such there is no reason to doubt the mathematical soundness of their constitutive equations. In fact Kelly has considered only one dimensional problems in the sense that there is only one velocity component. Such flows are highly artificial. It may not be possible to realise such flows in practice.

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