Chapter 1

Introduction

Let $\mathbb{X}$ be a complex Banach space with norm $\|.\|$ and $\mathcal{T}$ be a compact linear operator on $\mathbb{X}$. We are interested in numerically solving the following eigenvalue problem

$$\mathcal{T}u = \lambda u, \;\; \lambda \in \mathbb{C} - \{0\}, \;\; \|u\| = 1.$$

Many practical problems in science and engineering are formulated as eigenvalue problems of a compact linear integral operators $\mathcal{T}$ (cf., Chatelin [17]). As these problems, in general, can not be solved exactly, it becomes necessary to approximate the operator $\mathcal{T}$ by finite rank operators and then solve the original eigenvalue problem approximately. Let $\{\mathcal{T}_n\}$ be a sequence of finite rank operators on $\mathbb{X}$ approximating $\mathcal{T}$ in some sense, then the above eigenvalue problem can be approximated by the following approximated eigenvalue problem

$$\mathcal{T}_n u_n = \lambda_n u_n, \;\; \lambda_n \in \mathbb{C} - \{0\}, \;\; \|u_n\| = 1.$$

For the last many years, numerical solution of the eigenvalue problems have been attracted much attention. The Galerkin, Petrov-Galerkin, collocation, Nyström and degenerate kernel methods are the commonly used methods for the approximation of eigenelements of a compact integral operator. The analysis for the convergence of the Galerkin, Petrov-Galerkin, collocation, Nyström and degenerate kernel methods are well documented in Ahues et al. [1], Chatelin [17], Nelakanti [45], [46] and Osborn [47]. Error bounds for the eigenvalues of a compact integral operator with various kernels can be found in Burgmeier and Scott [13].

The general framework for the numerical solution of the eigenvalue problem of a compact integral operator was discussed in Atkinson [6]. The convergence of the Nyström

method has received a definite treatment in Anselone [5], by the introduction of the collectively compact convergence concept. The convergence of eigenelements of a compact integral operator in Nyström method were discussed in Atkinson [7]. Degenerate kernel method for the eigenvalue problems of a compact integral operator was discussed in Schafer [51] and Nelakanti [46]. Projection methods have been used for a long time. Their abstract treatment goes back at least to Kantorovitch [32], Kantorovitch and Akilov [33]. The convergence of projection methods has been widely studied in Kantorovitch and Akilov [33], Krasnoselskii et al. [36]. A very comprehensive treatment of methods for the eigenvalue problem of integral operators were given in Baker [10], Chatelin [17] and Ahues et al. [1]. The general framework for projection and Nyström methods for the approximations of eigenelements and their convergence rates for compact integral operator were given in Osborn [47] and Chatelin [17] under collectively compact convergence and norm convergence. In Nair [44] and Ahues et al. [1], existence of the eigenelements and their convergence rates were given under the ν-convergence which is weaker than norm convergence. The approximation of a eigenelements of a compact integral operator with weakly singular kernels were studied in Chen et al. [21]. The superconvergence results for the iterated eigenvector first noted by Sloan [53], [52] for the projection methods (orthogonal projection) and in de Boor and Swartz [23], [24] for interpolatory projection at Gaussian points. The superconvergence results for the iterated eigenvector of a compact integral operator with Greens kernels were studied in Chatelin [15], [16], [17]. Multigrid methods for the eigenvalue problem can be found in Hackbusch [31].

The influence of approximate quadrature in the projection methods for integral equations were studied in Atkinson and Potra [9], Atkinson and Bogomolny [8], Golberg [28] and Chen et al. [22]. In Kulkarni and Nelakanti [40], these approximate quadratures for the projection methods were extended to the eigenvalue problem of a compact integral operator and obtained the superconvergence results for the eigenelements. In Chen et al. [18], these superconvergence results for the eigenvalues were further improved by using Richardson extrapolation in iterated discrete methods.

Iterative refinement techniques improve the accuracy of eigenelements of a compact integral operator. Let λ be a simple eigenvalue. In the iterative refinement technique, an eigenvalue problem of relatively small size associated with a finite rank operator $\mathcal{T}_{n_0}$

is solved. The eigenelements of $\mathcal{T}_{n_0}$ then provide the initial iterates for the refinement schemes. Each iteration step involves only a solution of a system of equations. The iteration schemes such as Rayleigh-Schrodinger scheme, elementary iteration and double iteration were extensively studied in the literature (cf., Ahues et al. [1], Chatelin [17], Deshpande and Limaye [25], Kulkarni and Limaye [39]). Iterative refinement schemes for a cluster of eigenvalues were studied in Ahues et al. [1].

1.1 Motivation and overview of the work presented in this thesis

Let $\mathbb{X} = L^2(\Omega)$ or $L^\infty(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^d$, $d \geq 1$, with norm $\|.\|$ and let $\mathcal{T}$ be the integral operator defined on $\mathbb{X}$. Consider the following eigenvalue problem: find $u \in \mathbb{X}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$\mathcal{T}u = \lambda u, \ \ \|u\| = 1. \tag{1.1}$$

Let $\{\mathbb{X}_n\}$ be a sequence of finite dimensional subspaces of $\mathbb{X}$. If $\mathcal{P}_n$ is a sequence of projections from $\mathbb{X}$ into $\mathbb{X}_n$ converging to the identity operator on $\mathbb{X}$, then $\mathcal{T}$ can be written as the following matrix form

$$\begin{bmatrix} \mathcal{T}_n^{LL} & \mathcal{T}_n^{HL} \\ \mathcal{T}_n^{LH} & \mathcal{T}_n^{HH} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_n\mathcal{T}\mathcal{P}_n & \mathcal{P}_n\mathcal{T}(\mathcal{I}-\mathcal{P}_n) \\ (\mathcal{I}-\mathcal{P}_n)\mathcal{T}\mathcal{P}_n & (\mathcal{I}-\mathcal{P}_n)\mathcal{T}(\mathcal{I}-\mathcal{P}_n) \end{bmatrix}. \tag{1.2}$$

If $\mathcal{P}_n$ chosen to be orthogonal projection, then in the standard Galerkin method $\mathcal{T}$ is approximated by $\mathcal{T}_n^{LL} = \mathcal{P}_n\mathcal{T}\mathcal{P}_n$, in the standard projection method $\mathcal{T}$ is approximated by sum of the operators $\mathcal{T}_n^{LL}$ and $\mathcal{T}_n^{HL}$, in the iterated Galerkin method $\mathcal{T}$ is approximated by sum of the operators $\mathcal{T}_n^{LL}$ and $\mathcal{T}_n^{LH}$, and in the M-Galerkin method $\mathcal{T}$ is approximated by the sum of the blocks $\mathcal{T}_n^{LL}$, $\mathcal{T}_n^{LH}$ and $\mathcal{T}_n^{HL}$. Thus we have the following approximation methods for the eigenproblem (1.1):

Projection method: find $u_n^p \in \mathbb{X}_n$ and $\lambda_n^p \in \mathbb{C} - \{0\}$ such that

$$\mathcal{P}_n\mathcal{T}u_n^p = \lambda_n^p u_n^p, \ \|u_n^p\| = 1. \tag{1.3}$$

Galerkin method: find $u_n^G \in \mathbb{X}_n$ and $\lambda_n^G \in \mathbb{C} - \{0\}$ such that

$$\mathcal{P}_n\mathcal{T}\mathcal{P}_n u_n^G = \lambda_n^G u_n^G, \ \|u_n^G\| = 1. \tag{1.4}$$

Iterated Galerkin method: find $u_n^S \in \mathbb{X}_n$ and $\lambda_n^S \in \mathbb{C} - \{0\}$ such that

$$\mathcal{T}\mathcal{P}_n u_n^S = \lambda_n^S u_n^S, \ \|u_n^S\| = 1. \tag{1.5}$$

M-Galerkin method:(Kulkarni [38], Chen et al. [19]) find $u_n^M \in \mathbb{X}$ and $\lambda_n^M \in \mathbb{C} - \{0\}$ such that

$$\mathcal{T}_n^M u_n^M = \lambda_n^M u_n^M, \ \|u_n^M\| = 1, \tag{1.6}$$

where $\mathcal{T}_n^M = \mathcal{P}_n\mathcal{T}\mathcal{P}_n + (\mathcal{I} - \mathcal{P}_n)\mathcal{T}\mathcal{P}_n + \mathcal{P}_n\mathcal{T}(\mathcal{I} - \mathcal{P}_n)$.
If $\mathcal{P}_n$ chosen to be an interpolatory projection then Galerkin, iterated Galerkin and M-Galerkin methods are referred, respectively, as collocation, iterated collocation and M-collocation methods.

Standard numerical treatment in the Galerkin method for the eigenvalue problem (1.1) is normally to discretize the operator $\mathcal{P}_n\mathcal{T}\mathcal{P}_n$ into a matrix and then solve the eigenvalue problem of a resulting matrix. The computed eigenelements of the matrix are considered as approximations of eigenelements of the compact linear integral operator. It is well known that the matrix resulting from a discretization of the operator $\mathcal{P}_n\mathcal{T}\mathcal{P}_n$ is a full matrix. Solving the eigenvalue problem of a full matrix requires a significant amount of computational effort. Hence, fast algorithms for such a problem are highly desirable.

In Chapter-2, we are interested in developing a truncated wavelet Galerkin method for solving the eigenvalue problem (1.1) of a compact linear integral operator $\mathcal{T}$ with a smooth kernel. The essence of these methods is to approximate the matrix by a sparse matrix and solve the eigenvalue problem of a sparse matrix. Such algorithms are called matrix truncation or matrix compression techniques. These matrix compression techniques lead to fast algorithms for solving the eigenvalue problems. We develop truncated wavelet Galerkin methods for finding the eigenelements of a compact integral operator $\mathcal{T}$ with a smooth kernel based on the matrix compression techniques developed in Maass et al. [42], Pereverzev [48]. We obtain error bounds for the eigenvalues and distance between the spectral subspaces. By iterating the eigenvectors, we also obtain the improved error bounds for the eigenvectors. Also computationally, we show that in the truncated matrix eigenvalue problem, the number of inner products need to be evaluated in generating the matrices is less in comparison to the Galerkin method. This makes the truncated wavelet Galerkin method computationally economic in comparison to the wavelet Galerkin

method for the eigenvalue problem (1.1). Further, we consider the elementary and double iterations using the truncated wavelet Galerkin method for the eigenvalue problem (1.1) and we obtain precise orders of convergence for the eigenvalue and eigenvector iterates. We show conclusively that the double iteration is superior to the elementary iteration. Also, we show that in both the elementary and double iterations, for every step of iteration, the matrix to be inverted remains same and sparse.

In practice, while solving the approximated eigenvalue problems (1.4) and (1.5), it becomes necessary to replace the integrals corresponding to the integral operator $\mathcal{T}$ and the inner products by a numerical quadrature formulae. This leads to discrete methods (cf., Atkinson and Bogomolny [8], Kulkarni and Nelakanti [40]). If $\mathcal{P}_n$ is chosen to be an orthogonal projection, the approximate eigenvalue problems (1.4) and (1.5) give rise to discrete Galerkin and iterated discrete Galerkin methods, respectively. If $\mathcal{P}_n$ is chosen to be an interpolatory projection, the approximate eigenvalue problems (1.4) and (1.5) give rise to discrete collocation and iterated discrete collocation methods, respectively. In Kulkarni and Nelakanti [40], discrete and iterated discrete projection (Galerkin and collocation) methods were discussed for the one dimensional eigenvalue problem (1.1) with a smooth kernel. By choosing the numerical quadrature rule sufficiently accurate, it was shown that the convergence rates were preserved for the eigenelements and they obtained the superconvergence results. In Chen et al. [20], Richardson extrapolation methods were discussed and obtained the improved error bounds for the eigenvalues in both iterated discrete Galerkin and iterated discrete collocation methods.

In Chapter-3, we consider the eigenvalue problem of a two-dimensional compact integral operator with a smooth kernel and we obtain superconvergence rates for both eigenvalues and gap between the spectral subspaces in discrete Galerkin (discrete collocation) and iterated discrete Galerkin (iterated discrete collocation) methods. Using discrete Richardson extrapolation, we further improve the convergence rates for the eigenvalues.

However, using piecewise polynomials as basis functions, the computation of a sufficiently accurate eigenvalues and eigenvectors may require the use of much finer partition of the domain Ω and hence the size of the matrix eigenvalue problems corresponding to the eigenvalue problems (1.4) and (1.5) become very large as the norm of the partition becomes smaller. On the other hand, it is well known that the matrices in these matrix eigenvalue

problems are dense and it requires huge computation to generate these matrices. We are interested in the Galerkin and the collocation methods for the eigenvalue problem (1.1) using global polynomial basis functions rather than piecewise polynomial basis functions. Obviously, low degree polynomials imply small matrix eigenvalue problem, something which is highly desirable in practical computations. The Legendre polynomial basis functions are easy to generate recursively and they have the nice property of orthogonality. Hence Legendre basis functions are less expensive computationally in solving the matrix eigenvalue problem in comparison to piecewise polynomial basis functions.

In Chapter-4, we use Legendre polynomials as basis functions to find the approximate eigenelements in Galerkin and collocation methods. We choose $\mathbb{X}_n$ as a sequence of Legendre polynomial subspaces of $\mathbb{X}$ of degree $\leq n$ and $\mathcal{P}_n$ is either orthogonal or interpolatory projection operator from $\mathbb{X}$ into $\mathbb{X}_n$. It is known that $\|\mathcal{P}_n\|_\infty$ is unbounded. Therefore, there exists at least one $u \in \mathcal{C}[-1,1]$ such that $\|\mathcal{P}_n u - u\|_\infty \nrightarrow 0$ as $n \to \infty$. However, we obtain superconvergence results for the eigenelements in both L^2 and infinity norm.

To improve the error bounds further, in Chapter-5 we consider M-Galerkin and M-collocation methods (1.6) for the eigenvalue problem (1.1) using Legendre polynomial basis. We obtain the improved superconvergence results for the eigenelements in both L^2 and infinity norm.

The fine spectrum of various operators on sequence space c and c_0 were studied by Wenger [54], Rhoades [50], Reade [49], Yildirim [56]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c is determined by Altay and Basar [3]. The fine spectrum of the generalized difference operator $B(r,s)$ for constant real numbers r,s over the sequence spaces c_0 and c is established by Altay and Basar [4], which has been further generalized to difference operator $B(r,s,t)$ over sequence spaces c_0 and c is by Furkan et al. [26], where r,s,t are taken as real scalars.

The fine spectra of various operators on sequence space l_p is studied by Gonzàlez [30], Cartlidge [14], Akhmedov and Basar [2], Bilgic and Furkan [12]. The fine spectra of difference operator Δ over the sequence spaces l_1 and bv is studied by Kayaduman and Furkan [35]; later the fine spectrum of the generalized difference operator $B(r,s)$ over sequence spaces l_1 and bv is established by Furkan et al. [27]. The fine spectrum of the generalized difference operator $B(r,s,t)$ over sequence spaces l_1 and bv is established by

Bilgic and Furkan [11], where r, s, t are taken as scalars.

Chapte-6 & 7 are the continuation of the previous works discussed for fine spectrum, where we give the characterization of spectrum and fine spectrum of the generalized second order forward difference operator Δ^2_{uvw} for various real sequences $u = (u_k)$, $v = (v_k)$ and $w = (w_k)$ under certain restrictions over the sequence space c_0 and l_1. It is easy to verify that by choosing suitably u, v and w sequences, i.e., for suitable Δ^2_{uvw} one can get easily the operators such as $B(r, s, t)$ and Δ^2. If $u = (r), v = (s)$ and $w = (t)$, then the operator Δ^2_{uvw} reduces to $B(r, s, t)$. If $u = (1)$, $v = (-2)$ and $w = (1)$ are constant sequences, then the operator Δ^2_{uvw} reduces to second order forward difference operator Δ^2. Thus, the results of fine spectrum of Δ^2_{uvw} unifies the corresponding results of many operators whose matrix representation is a triple-band matrix.

1.2 Preliminaries

Let $\mathbb{X}$ be a complex Banach space and $\mathcal{BL}(\mathbb{X})$ denote the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ along with the operator norm, that is, for $\mathcal{T}$ in $\mathcal{BL}(\mathbb{X})$,

$$\|\mathcal{T}\| = \sup\{\|\mathcal{T}x\| : \ \|x\| \leq 1, \ x \in \mathbb{X}\}.$$

For an operator $\mathcal{T}$ in $\mathcal{BL}(\mathbb{X})$, we define its resolvent set by

$$\rho(\mathcal{T}) = \{z \in \mathbb{C} : (\mathcal{T} - z\mathcal{I})^{-1} \in \mathcal{BL}(\mathbb{X})\},$$

and its spectrum by $\sigma(\mathcal{T}) = \mathbb{C} \setminus \rho(\mathcal{T})$.

The spectral radius of the operator $\mathcal{T}$ is defined by $r_\sigma(\mathcal{T}) = \sup\{|\lambda| : \ \lambda \in \sigma(\mathcal{T})\}$. Since $r_\sigma(\mathcal{T}) \leq \|\mathcal{T}\|$ and the resolvent set $\rho(\mathcal{T})$ is open, it follows that $\sigma(\mathcal{T})$ is compact. For $\mathcal{T} \in \mathcal{BL}(\mathbb{X})$, we denote range space and null space of $\mathcal{T}$, respectively, by

$$\mathcal{R}(\mathcal{T}) = \{\mathcal{T}x : \ x \in \mathbb{X}\} \text{ and } \mathcal{N}(\mathcal{T}) = \{x \in \mathbb{X} : \mathcal{T}x = 0\}.$$

For $z \in \rho(\mathcal{T})$, let

$$R(z) = (\mathcal{T} - z\mathcal{I})^{-1}.$$

Let Γ be a simple closed Jordan curve in $\rho(\mathcal{T})$, then $Int\ \Gamma$ denote the bounded component of $\mathbb{C} \setminus \Gamma$. Let

$$\mathcal{P}^S = -\frac{1}{2\pi i} \int_\Gamma R(z)\ \mathrm{d}z,$$

denotes the spectral projection associated with $\mathcal{T}$ and $\Lambda = \sigma(\mathcal{T}) \cap Int\ \Gamma$.

Definition 1.1. *A subspace $\mathbb{Y}$ of $\mathbb{X}$ is called invariant under $\mathcal{T}$ if $\mathcal{T}(\mathbb{Y}) \subset \mathbb{Y}$.*

We denote by $\mathcal{T}_{|_{\mathbb{Y}}}$ the restriction of the operator $\mathcal{T}$ from $\mathbb{Y}$ into $\mathbb{Y}$. We quote the following Theorem from Chatelin [17] and Kato [34].

Theorem 1.1. (Spectral Decomposition Theorem) *Let $\mathcal{T} \in \mathcal{BL}(\mathbb{X})$ and $\Gamma \subset \rho(\mathcal{T})$. Let $\mathcal{P}^S$ be the spectral projection associated with $\mathcal{T}$ and $\sigma(\mathcal{T}) \cap Int\ \Gamma$. Then $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{N}(\mathcal{P}^S)$ are invariant under $\mathcal{T}$. Moreover, $\sigma(\mathcal{T})$ is the disjoint union of $\sigma(\mathcal{T}_{|\mathcal{R}(\mathcal{P}^S)})$ and $\sigma(\mathcal{T}_{|\mathcal{N}(\mathcal{P}^S)})$. In fact*

$$\sigma(\mathcal{T}_{|\mathcal{R}(\mathcal{P}^S)}) = \sigma(\mathcal{T}) \cap Int\ \Gamma, \quad \sigma(\mathcal{T}_{|\mathcal{N}(\mathcal{P}^S)}) = \sigma(\mathcal{T}) \cap Ext\ \Gamma.$$

Now let $\lambda \neq 0$ be an isolated point of $\sigma(\mathcal{T})$ and Γ be the positively oriented closed curve around λ, which isolates λ from zero and from rest of the spectrum of $\mathcal{T}$, i.e., $\sigma(\mathcal{T}) \cap Int\ \Gamma = \{\lambda\}$. Let $\mathcal{P}^S$ be the spectral projection associated with $\mathcal{T}$ and λ. If rank $\mathcal{P}^S = m < \infty$, then λ is an eigenvalue of $\mathcal{T}$ and m is known as its algebraic multiplicity. Let $\mathcal{D} = (\mathcal{T} - \lambda\mathcal{I})\mathcal{P}^S$. If ℓ is the least positive integer such that $\mathcal{D}^\ell = 0$, then ℓ is called the ascent of λ. In fact, if ℓ is the ascent of λ, then

$$\mathcal{N}((\mathcal{T} - \lambda\mathcal{I})^\ell) = \mathcal{N}((\mathcal{T} - \lambda\mathcal{I})^{\ell+1}) \quad \text{and} \quad \mathcal{R}(\mathcal{P}^S) = \mathcal{N}((\mathcal{T} - \lambda\mathcal{I})^\ell).$$

Let

$$\mathcal{S} = \frac{1}{2\pi i} \int_\Gamma \frac{(\mathcal{T} - z\mathcal{I})^{-1}}{z - \lambda}\, \mathrm{d}z,$$

be the reduced resolvent associated with $\mathcal{T}$ and λ. Then $\mathcal{S}$ satisfies the following equations

$$\begin{aligned} (\mathcal{T} - \lambda\mathcal{I})\mathcal{S} &= \mathcal{S}(\mathcal{T} - \lambda\mathcal{I}) = \mathcal{I} - \mathcal{P}^{\mathcal{S}}, \\ \mathcal{S}\mathcal{P}^S &= \mathcal{P}^S\mathcal{S} = 0. \end{aligned}$$

Spectral approximation

Consider the eigenvalue problem

$$\mathcal{T}u = \lambda u, \ \ 0 \neq \lambda \in \mathbb{C}, \ \ \|u\| = 1. \tag{1.7}$$

As the above problem, in general, can not be solved exactly, we consider approximate solutions. For this purpose let $(\mathcal{T}_n)$ be a sequence in $\mathcal{BL}(\mathbb{X})$ approximating $\mathcal{T}$. We describe below various modes of convergence.

The classical convergence is **Norm convergence:** $\|\mathcal{T} - \mathcal{T}_n\| \to 0$, as $n \to \infty$.

When $\mathcal{T}$ is compact, the following mode of convergence which is weaker than the norm convergence is considered in Ahues et al. [1].

ν-Convergence: (cf., Ahues et al. [1]) $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$ if

$$\|\mathcal{T}_n\| \leq C, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}\| \to 0, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}_n\| \to 0 \text{ as } n \to \infty.$$

If $\mathcal{T}_n$ converges to $\mathcal{T}$ in norm, then it is clear that $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$.

Lemma 1.1. (Anselone [5]) *Let $\mathbb{X}$ be a Banach space and $S \subset \mathbb{X}$ be a relatively compact set. Assume that $\mathcal{T}$ and $\mathcal{T}_n$ are bounded linear operators on $\mathbb{X}$ satisfying $\|\mathcal{T}_n\| \leq C$ for all $n \in \mathbb{N}$, and for each $x \in \mathbb{X}$,*

$$\|\mathcal{T}_n x - \mathcal{T} x\| \to 0 \text{ as } n \to \infty,$$

where C is a constant independent of n. Then $\|\mathcal{T}_n x - \mathcal{T} x\| \to 0$ as $n \to \infty$ uniformly for all $x \in S$.

For the rest of the chapter we assume that $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$ or $\mathcal{T}_n$ converges to $\mathcal{T}$ in norm. The following result is proved in Nair [44].

Lemma 1.2. *Let E be a closed subset of $\rho(\mathcal{T}) - \{0\}$ and $\delta = \min\{|z| : z \in E\}$. Then there is a constant C_1 such that*

$$\max_{z \in E} \|R(z)\| \leq C_1.$$

Let n_0 be a positive integer such that

$$\|(\mathcal{T} - \mathcal{T}_n)^2\| < \delta^2 \text{ and } C_1 \|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}_n\| \leq \frac{\delta}{2},$$

for all $n \geq n_0$. Then $E \subset \rho(\mathcal{T}_n)$ and

$$\max_{z \in E} \|R_n(z)\| \leq 2C_1 \left[1 + \frac{\|\mathcal{T} - \mathcal{T}_n\|}{\delta}\right] \leq C_2,$$

for some constant C_2 and all $n \geq n_0$.

Letting $E = \Gamma$ in Lemma 1.2, we see that $\Gamma \subset \rho(\mathcal{T}_n)$ for all large n. For such n, let

$$\mathcal{P}_n^S = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{T}_n - z\mathcal{I})^{-1}\,\mathrm{d}z,$$

denote the spectral projection associated with $\mathcal{T}_n$ and $\sigma(\mathcal{T}_n) \cap Int\,\Gamma$. We assume rank of $\mathcal{P}^S = m < \infty$, i.e., λ is an eigenvalue of $\mathcal{T}$ with algebraic multiplicity m.

Proposition 1.1. (Ahues et al. [1]) *For sufficiently large n, the following hold*

$$\begin{aligned}
&\|\mathcal{P}^S - \mathcal{P}_n^S\| \le C\|\mathcal{T} - \mathcal{T}_n\|,\\
&\|(\mathcal{P}^S - \mathcal{P}_n^S)\mathcal{P}^S\| \le C\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|,\\
&\|(\mathcal{P}^S - \mathcal{P}_n^S)\mathcal{P}_n^S\| \le C\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}_n\|,
\end{aligned}$$

where C is a constant independent of n.

We quote the following result from Ahues et al. [1], pp. 86.

Proposition 1.2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two projections on $\mathbb{X}$. If $r_\sigma(\mathcal{P}_1 - \mathcal{P}_2) < 1$, then*

$$rank\,\mathcal{P}_1 = rank\,\mathcal{P}_2.$$

Now since

$$r_\sigma(\mathcal{P}^S - \mathcal{P}_n^S) \le C\|(\mathcal{P}^S - \mathcal{P}_n^S)\| \to 0,$$

and

$$r_\sigma((\mathcal{P}^S - \mathcal{P}_n^S)^2) \le \|(\mathcal{P}^S - \mathcal{P}_n^S)^2\| \le \|(\mathcal{P}^S - \mathcal{P}_n^S)\mathcal{P}^S\| + \|(\mathcal{P}^S - \mathcal{P}_n^S)\mathcal{P}_n^S\| \to 0,$$

as $n \to \infty$, by Proposition 1.2 we obtain

$$\text{rank } \mathcal{P}_n^S = \text{rank } \mathcal{P}^S = m.$$

Since rank $\mathcal{P}_n^S = m$ for all large n, the Spectral Decomposition Theorem 1.1 shows that, $\Lambda_n = \sigma(\mathcal{T}_n) \cap Int\,\Gamma$ consists of m eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m}$ of $\mathcal{T}_n$, counted according to their algebraic multiplicities. Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,m}}{m},$$

denote their arithmetic mean.

To discuss the closeness of eigenvectors, we define the gap between the closed subspaces. For nonzero closed subspaces $\mathbb{Y}_1$ and $\mathbb{Y}_2$ of $\mathbb{X}$, let

$$\delta_p(\mathbb{Y}_1, \mathbb{Y}_2) = \sup\{\text{dist}(y, \mathbb{Y}_2) : y \in \mathbb{Y}_1, \ \ \|y\|_{L^p} = 1\}, \ 1 \leq p \leq \infty,$$

then

$$\hat{\delta}_p(\mathbb{Y}_1, \mathbb{Y}_2) = \ \max\{\delta_p(\mathbb{Y}_1, \mathbb{Y}_2), \delta_p(\mathbb{Y}_2, \mathbb{Y}_1)\},$$

is known as the gap between $\mathbb{Y}_1$ and $\mathbb{Y}_2$.

The following Lemmas give the gap between the spectral subspaces, i.e., nearness of the eigenvectors (cf., Ahues et al. [1], Chatelin [17], Nelakanti [46], Osborn [47]).

Lemma 1.3. ([1], [46] and [47]) *Let $\mathcal{T}_n$ be the sequence of operators such that $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ or $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$. Let $\mathcal{P}^S$ and $\mathcal{P}_n^S$ be the spectral projections of $\mathcal{T}$ and $\mathcal{T}_n$, respectively, associated with their corresponding spectra inside Γ. Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be their ranges. Then for sufficiently large n, there exists a constant C independent of n such that*

$$\hat{\delta}(\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq C \sup\{\|(\mathcal{T} - \mathcal{T}_n)u\| : \ u \in \mathcal{R}(\mathcal{P}^S)\},$$
$$\delta(\mathcal{T}\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq C \sup\{\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)u\| : \ u \in \mathcal{R}(\mathcal{P}^S)\}.$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, $\tilde{u}_n = \mathcal{T}u_n$, we have

$$\|u_n - \mathcal{P}^S u_n\|_\infty \leq C \sup\{\|(\mathcal{T} - \mathcal{T}_n)u\| : \ u \in \mathcal{R}(\mathcal{P}^S)\},$$
$$\|\tilde{u}_n - \mathcal{P}^S \tilde{u}_n\|_\infty \leq C \sup\{\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)u\| : \ u \in \mathcal{R}(\mathcal{P}^S)\}.$$

Lemma 1.4. ([1], [17], [47]) *Let $\mathcal{T}_n$ be the sequence of operators such that $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ or $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$. Let $\mathcal{P}^S$ and $\mathcal{P}_n^S$ be the spectral projections of $\mathcal{T}$ and $\mathcal{T}_n$, respectively, associated with their corresponding spectra inside Γ. Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be their ranges. Then for sufficiently large n, there exists a constant C independent of n such that*

$$\hat{\delta}(\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \ \ \leq \ \ C\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|.$$

In particular for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|u_n - \mathcal{P}^S u_n\| \ \ \leq \ \ C\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|.$$

Lemma 1.5. ([46]) *Let $\mathcal{T}_n$ be the sequence of operators such that $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ or $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$. Let $\mathcal{P}^S$ and $\mathcal{P}_n^S$ be the spectral projections of $\mathcal{T}$ and $\mathcal{T}_n$, respectively, associated with their corresponding spectra inside Γ. Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be their ranges. Then for sufficiently large n, there exists a constant C independent of n such that*

$$\delta(\mathcal{T}\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq C\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)\mathcal{T}_n\|.$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|\mathcal{T}u_n - \mathcal{P}^S\mathcal{T}u_n\| \leq C\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)\mathcal{T}_n\|.$$

The following Lemmas give the error bounds for eigenvalues (cf., Kulkarni and Nelakanti [41], Nelakanti [45] [46]).

Lemma 1.6. ([41], [45]) *Let λ be an eigenvalue of $\mathcal{T}$ with algebraic multiplicity m and ascent ℓ. For all large n, the map $A_n = \mathcal{P}_n^S|_{\mathcal{R}(\mathcal{P}^S)} : \mathcal{R}(\mathcal{P}^S) \to \mathcal{R}(\mathcal{P}_n^S)$ is invertible and $\|A_n^{-1}\| \leq 2$, and the following hold*

$$|\lambda - \hat{\lambda}_n| \leq C \sup\{\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)u\| : u \in \mathcal{R}(\mathcal{P}^S)\},$$
$$|\lambda - \lambda_{n,j}|^\ell \leq \alpha_n \sup\{\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)u\| : u \in \mathcal{R}(\mathcal{P}^S)\},$$

where

$$\alpha_n = \sum_{k=0}^{l-1} \|\lambda\mathcal{I}|_{\mathcal{R}(\mathcal{P}^S)} - A_n^{-1}\mathcal{T}_n A_n\|^{l-1-k} \|\lambda\mathcal{I}|_{\mathcal{R}(\mathcal{P}^S)} - \mathcal{T}|_{\mathcal{R}(\mathcal{P}^S)}\|^k,$$

and C is a bounded constant independent of n.

Lemma 1.7. ([45]) *If $\mathbb{X}$ is Banach space, $\mathcal{T}, \mathcal{T}_n \in \mathcal{BL}(\mathbb{X})$ for $n \in \mathbb{N}$ and $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ or $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$, then for sufficiently large n, there exists a constant C independent of n such that for $j = 1, 2, \ldots, m$,*

$$|\lambda - \hat{\lambda}_n| \leq C\|\mathcal{T}_n(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|,$$
$$|\lambda - \lambda_{n,j}|^\ell \leq C\|\mathcal{T}_n(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|,$$

where C is a constant independent of n.

Iterative refinement schemes:

Let $\mathbb{X}^*$ denote the adjoint space of $\mathbb{X}$, i.e., the space of all conjugate-linear functionals on $\mathbb{X}$. We shall use the following notation:

$$\langle x,\, f\rangle = \overline{f(x)},\ x \in \mathbb{X},\ f \in \mathbb{X}^*.$$

We define the adjoint operator $\mathcal{T}^* \in \mathcal{BL}(\mathbb{X}^*)$ as

$$\mathcal{T}^* f = f\mathcal{T},\ \text{for all } f \in \mathbb{X}^*.$$

Then

$$\langle x,\, \mathcal{T}^* f\rangle = \langle \mathcal{T}x,\, f\rangle,\ \text{ for all } x \in \mathbb{X},\ f \in \mathbb{X}^*.$$

Now we assume that λ be a simple eigenvalue of $\mathcal{T}$ and $\mathcal{P}^S$ be the spectral projection associated with $\mathcal{T}$ and λ. Then rank $\mathcal{P}^S = 1$.
Since $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$, there exists n_0 such that for $n \geq n_0$, $\Gamma \subset \rho(\mathcal{T}_n)$,

$$\sup\{\|(\mathcal{T}_n - z\mathcal{I})^{-1}\| : z \in \Gamma,\quad n \geq n_0\} < \infty,$$

and the spectrum of $\mathcal{T}_n$ inside Γ consists of a simple eigenvalue λ_n. Hence $\|\mathcal{P}_n^S\| \leq p$ and

$$\text{rank } \mathcal{P}^S = \text{rank } \mathcal{P}_n^S = 1.$$

Let

$$\mathcal{S}_n = \frac{1}{2\pi i}\int_\Gamma \frac{(\mathcal{T}_n - z\mathcal{I})^{-1}}{z - \lambda_n}\,\mathrm{d}z,$$

be the reduced resolvent associated with $\mathcal{T}_n$ and λ_n. Then we have

$$\mathcal{S}_n(\mathcal{T}_n - \lambda_n\mathcal{I}) = (\mathcal{T}_n - \lambda_n\mathcal{I})\mathcal{S}_n = \mathcal{I} - \mathcal{P}_n^S,\quad \mathcal{S}_n\mathcal{P}_n^S = \mathcal{P}_n^S\mathcal{S}_n = 0.$$

Let

$$\mathcal{T}_n^* u_n^* = \overline{\lambda}_n u_n^*,\quad \langle u_n,\, u_n^*\rangle = 1.$$

The spectral projection $\mathcal{P}_n^S$ is then given by

$$\mathcal{P}_n^S x = \langle x,\, u_n^*\rangle u_n,\quad x \in \mathbb{X}.$$

Thus $\|\mathcal{P}_n^S\| = \|u_n^*\|$. Since $\|(\mathcal{P}_n^S - \mathcal{P}^S)\mathcal{P}^S\| \to 0$, it follows that $\|\mathcal{P}_n^S u - u\| \to 0$. Hence

$$\|\langle u,\, u_n^*\rangle u_n\| = |\langle u,\, u_n^*\rangle| \to \|u\| = 1,$$

and for sufficiently large n, $\frac{1}{2} \leq |\langle u,\, u_n^* \rangle|$.

We define

$$u_{(n)} = \frac{u}{\langle u,\, u_n^* \rangle}.$$

Then $u_{(n)}$ is an eigenvector of $\mathcal{T}$ associated with λ which satisfies $\langle u_{(n)},\, u_n^* \rangle = 1$.

Note that for all n large enough, $\|u_{(n)}\| \leq 2$. We consider the following two iterative refinement schemes (cf., Ahues et al. [1], Deshpande and Limaye [25], Kulkarni and Limaye [39]).

Elementary or fixed point iteration:

$$\begin{aligned}
&\lambda_n^{(0)} = \lambda_n, \quad u_n^{(0)} = u_n, \quad \text{and for} \quad k = 1, 2, \ldots \\
&\lambda_n^{(k)} = \langle \mathcal{T} u_n^{(k-1)},\, u_n^* \rangle, \\
&u_n^{(k)} = u_n^{(k-1)} + \mathcal{S}_n(\lambda_n^{(k)} u_n^{(k-1)} - \mathcal{T} u_n^{(k-1)}).
\end{aligned}$$

The following error bounds are well known (cf., Kulkarni and Nelakanti [41]).

Lemma 1.8. ([41]) *Let $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$ or $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ and $\lambda_n^{(k)}$, $u_n^{(k)}$ be the iterates in the elementary iteration. Then there is a positive integer n_1 such that for all $n \geq n_1$*

$$\begin{aligned}
|\lambda_n - \lambda| &\leq \frac{4p}{|\lambda|^2} \|\mathcal{T}_n(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|, \\
\|u_n - u_{(n)}\| &\leq \alpha \|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|.
\end{aligned}$$

Also, for $k = 1, 2, \ldots$

$$\begin{aligned}
|\lambda_n^{(k)} - \lambda| &\leq \frac{2p}{|\lambda|} \alpha \|\mathcal{T}_n(\mathcal{T} - \mathcal{T}_n)\| \|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\| (\beta \|\mathcal{T} - \mathcal{T}_n\|)^{k-1}, \\
\|u_n^{(k)} - u_{(n)}\| &\leq \alpha \|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\| (\beta \|(\mathcal{T} - \mathcal{T}_n)\|)^k,
\end{aligned}$$

where α and β are constants independent of n and k.

Double or modified fixed point iteration:

$$\begin{aligned}
&\mu_n^{(0)} = \lambda_n, \quad \psi_n^{(0)} = u_n \ \text{ and for } \quad k = 1, 2, \ldots \\
&\lambda_n^{(k)} = \langle \mathcal{T} \psi_n^{(k-1)},\, u_n^* \rangle, \\
&u_n^{(k-1)} = \frac{\mathcal{T} \psi_n^{(k-1)}}{\lambda_n^{(k)}}, \\
&\mu_n^{(k)} = \langle \mathcal{T} u_n^{(k-1)},\, u_n^* \rangle, \\
&\psi_n^{(k)} = u_n^{(k-1)} + \mathcal{S}_n(\mu_n^{(k)} u_n^{(k-1)} - \mathcal{T} u_n^{(k-1)}).
\end{aligned}$$

Lemma 1.9. ([41]) *Let $\mathcal{T}_n \xrightarrow{\nu} \mathcal{T}$ or $\|\mathcal{T} - \mathcal{T}_n\| \to 0$ and let $\mu_n^{(k)}$, $\psi_n^{(k)}$ be the iterates in the double iteration. Then there is a positive integer n_1 such that for all $n \geq n_1$ and for $k = 0, 1, 2, \ldots$*

$$|\mu_n^{(k)} - \lambda| \leq C\|\mathcal{T}_n(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|(\beta\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|)^k,$$
$$\|\psi_n^{(k)} - u_{(n)}\| \leq (\beta\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}\|)^{k+1},$$

where C and β are constants independent of n and k.

Remark 1.1. If $\mathcal{T}_n$ is of rank n, then the eigenvalue problem

$$\mathcal{T}_n u_n = \lambda_n u_n,$$

reduces to a matrix eigenvalue problem of size n. The computation of $y = \mathcal{S}_n x$ can be reduced to a solution of a system of equations of size n:

$$\begin{aligned}(\mathcal{T}_n - \lambda_n \mathcal{I})y &= x,\\ \mathcal{P}_n^S y &= 0.\end{aligned}$$

Thus at each step of the elementary iteration we need to solve a system of size n but the coefficient matrix is the same at each step. On the other hand, for double iteration in addition to the computations in the elementary iteration, we need to compute $\lambda_n^{(k)} = \langle \mathcal{T}\psi_n^{(k-1)}, u_n^* \rangle$ and $\dfrac{\mathcal{T}\psi_n^{(k-1)}}{\lambda_n^{(k)}}$.

We set the following notations:

Consider the following compact integral operator defined on $\mathbb{X} = L^2[a, b]$ or $L^\infty[a, b]$

$$\mathcal{K}u(s) = \int_a^b K(s, t)u(t)\, \mathrm{d}t, \tag{1.8}$$

with a kernel $K(., .) \in \mathcal{C}([a, b] \times [a, b])$.

For $r \geq 0$, let $\mathcal{C}^r[a, b]$ denote the space of r times continuously differentiable functions. If $K(., .) \in \mathcal{C}^r([a, b] \times [a, b])$, then $\mathcal{R}(\mathcal{K}) \in \mathcal{C}^r[a, b]$ and we denote

$$K^{(i,j)}(s, t) = \frac{\partial^{i+j}}{\partial s^i \partial t^j} K(s, t),$$

and

$$\|K\|_{r,\infty} = \max\{\|K^{(i,j)}(., .)\|_\infty : 0 \leq i \leq r,\ 0 \leq j \leq r\}.$$

Set $K_s(t) = K(s,t)$, for $s,t \in [a,b]$. Note that for $K(s,.) \in \mathcal{C}^j([a,b]),\ j = 0,1,2,\ldots,r$ and $u \in \mathcal{C}[a,b]$, we have

$$\begin{aligned} |(\mathcal{K}u)^{(j)}(s)| &= \Big|\int_a^b \frac{\partial^j}{\partial s^j}K(s,t)u(t)\,\mathrm{dt}\Big| \\ &\leq \max_{s,t\in[a,b]}\Big|\frac{\partial^j}{\partial s^j}K(s,t)\Big|\int_a^b |u(t)|\,\mathrm{dt} \\ &\leq (b-a)^{1/2}\|K\|_{j,\infty}\|u\|_{L^2} \leq (b-a)\|K\|_{j,\infty}\|u\|_\infty. \end{aligned}$$

From this, it follows that for $j = 0,1,2,\ldots,r$,

$$\|(\mathcal{K}u)^{(j)}\|_\infty \leq (b-a)^{1/2}\|K\|_{j,\infty}\|u\|_{L^2} \leq (b-a)\|K\|_{j,\infty}\|u\|_\infty, \tag{1.9}$$

and

$$\begin{aligned} \|(\mathcal{K}u)^{(j)}\|_{L^2} &\leq (b-a)^{1/2}\|(\mathcal{K}u)^{(j)}\|_\infty \\ &\leq (b-a)\|K\|_{j,\infty}\|u\|_{L^2} \leq (b-a)^{3/2}\|K\|_{j,\infty}\|u\|_\infty. \end{aligned} \tag{1.10}$$

Let $\mathcal{K}$ be the compact linear integral operator defined by (**??**) and λ be the simple eigenvalue of $\mathcal{K}$. Let $\mathcal{K}_n$ be a sequence of operators such that $\mathcal{K}_n \xrightarrow{\nu} K$ or $\|\mathcal{K} - \mathcal{K}_n\| \to 0$, as $n \to \infty$. Let $\mathcal{P}^{\mathcal{S}}$ and $\mathcal{P}_n^{\mathcal{S}}$ are the spectral projections of $\mathcal{K}$ and $\mathcal{K}_n$, respectively. Let $\mathcal{R}(\mathcal{P}^{\mathcal{S}})$ and $\mathcal{R}(\mathcal{P}_n^{\mathcal{S}})$ are ranges of the spectral projections $\mathcal{P}^S$ and $\mathcal{P}_n^{\mathcal{S}}$, respectively, then we prove the following Theorem.

Theorem 1.2. *Let $\mathbb{X}$ be a Banach space and let $\mathcal{K}, \mathcal{K}_n \in \mathcal{BL}(\mathbb{X})$ such that $\mathcal{K}_n \xrightarrow{\nu} K$ or $\|\mathcal{K} - \mathcal{K}_n\| \to 0$, as $n \to \infty$. Let $\mathcal{R}(\mathcal{P}^{\mathcal{S}})$ and $\mathcal{R}(\mathcal{P}_n^{\mathcal{S}})$ are ranges of the spectral projections $\mathcal{P}^S$ and $\mathcal{P}_n^{\mathcal{S}}$, respectively. Then for any $u_n \in \mathcal{R}(\mathcal{P}_n^{\mathcal{S}})$, we have*

$$\|\mathcal{K}u_n - \mathcal{P}^{\mathcal{S}}\mathcal{K}u_n\|_{j,\infty} = \max_{s\in[-1,1]}|\langle(\mathcal{K}-\mathcal{K}_n)u_n,\, \ell_s^{(j)}\rangle|,\quad j = 0,1,2,\ldots,$$

where $(\mathcal{K}^* - \bar{\lambda}_n)(\mathcal{I} - (\mathcal{P}^{\mathcal{S}})^*)\ell_s^{(j)} = (\mathcal{I} - (\mathcal{P}^{\mathcal{S}})^*)\bar{K}_s^{(j)}$.

Proof. For any $u_n \in \mathcal{R}(\mathcal{P}_n^{\mathcal{S}})$, we have (Chatelin [17]),

$$u_n - \mathcal{P}^{\mathcal{S}}u_n = \mathcal{S}(\lambda_n)(\mathcal{K} - \mathcal{K}_n)u_n, \tag{1.11}$$

where $\mathcal{S}(\lambda_n)$ denote the reduced resolvent of $\mathcal{K}$ at λ_n.
Now by applying $\mathcal{K}$ on both sides of the above equation (1.11) and using the fact that $\mathcal{K}\mathcal{P}^{\mathcal{S}} = \mathcal{P}^{\mathcal{S}}\mathcal{K}$, we obtain

$$\mathcal{K}u_n - \mathcal{P}^{\mathcal{S}}\mathcal{K}u_n = \mathcal{K}\mathcal{S}(\lambda_n)(\mathcal{K} - \mathcal{K}_n)u_n.$$

Hence

$$(\mathcal{K}u_n - \mathcal{P}^{\mathcal{S}}\mathcal{K}u_n)(s) = \int_a^b K(s,t)[\mathcal{S}(\lambda_n)(\mathcal{K}-\mathcal{K}_n)u_n](t)\ \mathrm{dt}.$$

For $j = 0, 1, 2, \ldots,$

$$\begin{aligned}(\mathcal{K}u_n - \mathcal{P}^{\mathcal{S}}\mathcal{K}u_n)^{(j)}(s) &= \int_a^b K^{(j)}(s,t)[\mathcal{S}(\lambda_n)(\mathcal{K}-\mathcal{K}_n)u_n](t)\ \mathrm{dt} \\ &= \langle \mathcal{S}(\lambda_n)(\mathcal{K}-\mathcal{K}_n)u_n,\ \bar{K}_s^{(j)}\rangle \\ &= \langle (\mathcal{K}-\mathcal{K}_n)u_n,\ (\mathcal{S}(\lambda_n))^*\bar{K}_s^{(j)}\rangle \\ &= \langle (\mathcal{K}-\mathcal{K}_n)u_n,\ \ell_s^{(j)}\rangle.\end{aligned}$$

In the above we have used the fact that $(\mathcal{S}(\lambda_n))^*\bar{K}_s^{(j)} = \ell_s^{(j)}$ is uniquely solvable. Since the kernel K_s is smooth, ℓ_s also smooth. Hence

$$\|\mathcal{K}u_n - \mathcal{P}^{\mathcal{S}}\mathcal{K}u_n\|_{j,\infty} = \max_{s\in[-1,1]} |\langle (\mathcal{K}-\mathcal{K}_n)u_n,\ \ell_s^{(j)}\rangle|.$$

This completes the proof. □

For any function $u \in \mathcal{C}^r[a,b]$, denote

$$\|u\|_{r,\infty} = \sum_{j=0}^{r} \|u^{(j)}\|_\infty,$$

where $u^{(j)}$ denotes the jth derivative of u.
Let $\mathbb{N} = \{1, 2, \ldots,\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $m \in \mathbb{N}$, we set $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$. Through out this thesis, we assume C as a generic constant.

1.2.1 Preliminaries for fine spectrum

Let $\mathbb{X}$ and $\mathbb{Y}$ be the Banach spaces and $\mathcal{T} : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. The adjoint $\mathcal{T}^*$ of $\mathcal{T}$ is a bounded linear operator on the dual space $\mathbb{X}^*$ of $\mathbb{X}$ defined by

$$(\mathcal{T}^* f)(x) = f(\mathcal{T}x) \text{ for all } f \in \mathbb{X}^* \text{ and } x \in \mathbb{X}.$$

Let $\mathbb{X} \neq \{0\}$ be a complex normed space and $\mathcal{T} : D(\mathcal{T}) \to \mathbb{X}$ be a linear operator with domain $D(\mathcal{T}) \subseteq \mathbb{X}$. With $\mathcal{T}$, we associate the operator $\mathcal{T}_\alpha = (\mathcal{T} - \alpha\mathcal{I})$, where α is a complex number and $\mathcal{I}$ is the identity operator on $D(\mathcal{T})$. The inverse of $\mathcal{T}_\alpha$ (if exists) is

denoted by $\mathcal{T}_\alpha^{-1}$, where $\mathcal{T}_\alpha^{-1} = (\mathcal{T} - \alpha\mathcal{I})^{-1}$ and known as the resolvent operator of $\mathcal{T}$. It is easy to verify that $\mathcal{T}_\alpha^{-1}$ is linear, if $\mathcal{T}_\alpha$ is linear. Since the spectral theory is concerned with many properties of $\mathcal{T}_\alpha$ and $\mathcal{T}_\alpha^{-1}$ which depend on α, so we are interested in the set of those α in the complex plane for which $\mathcal{T}_\alpha^{-1}$ exists or $\mathcal{T}_\alpha^{-1}$ is bounded or domain of $\mathcal{T}_\alpha^{-1}$ is dense in $\mathbb{X}$. For this, we need some definitions and known results given below which will be used in the sequel.

Definition 1.2. (Kreyszig [37], pp. 371) *Let $\mathbb{X} \neq \{0\}$ be a complex normed space and $\mathcal{T} : D(\mathcal{T}) \to \mathbb{X}$ be a linear operator with domain $D(\mathcal{T}) \subseteq \mathbb{X}$. A regular value of $\mathcal{T}$ is a complex number α such that*
(R1) $\mathcal{T}_\alpha^{-1}$ exists,
(R2) $\mathcal{T}_\alpha^{-1}$ is bounded,
(R3) $\mathcal{T}_\alpha^{-1}$ is defined on a set which is dense in $\mathbb{X}$.

Resolvent set $\rho(\mathcal{T}, \mathbb{X})$ of $\mathcal{T}$ is the set of all regular values α of $\mathcal{T}$. Its complement $\sigma(\mathcal{T}, \mathbb{X}) = \mathbb{C} \setminus \rho(\mathcal{T}, \mathbb{X})$ in the complex plane $\mathbb{C}$ is called spectrum of $\mathcal{T}$. The spectrum $\sigma(\mathcal{T}, \mathbb{X})$ is further partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual spectrum as follows:

Point Spectrum $\sigma_p(\mathcal{T}, \mathbb{X})$ is the set of all $\alpha \in \mathbb{C}$ such that $\mathcal{T}_\alpha^{-1}$ does not exists, i.e., condition $(R1)$ fails. The element of $\sigma_p(\mathcal{T}, \mathbb{X})$ is called eigenvalue of $\mathcal{T}$.

Continuous spectrum $\sigma_c(\mathcal{T}, \mathbb{X})$ is the set of all $\alpha \in \mathbb{C}$ such that conditions $(R1)$ and $(R3)$ hold but condition $(R2)$ fails, i.e., $\mathcal{T}_\alpha^{-1}$ exists, domain of $\mathcal{T}_\alpha^{-1}$ is dense in $\mathbb{X}$ but $\mathcal{T}_\alpha^{-1}$ is unbounded.

Residual Spectrum $\sigma_r(\mathcal{T}, \mathbb{X})$ is the set of all $\alpha \in \mathbb{C}$ such that $\mathcal{T}_\alpha^{-1}$ exists but do not satisfy conditions $(R3)$, i.e., domain of $\mathcal{T}_\alpha^{-1}$ is not dense in $\mathbb{X}$. The condition $(R2)$ may or may not holds good.

Goldberg's classification of operator $\mathcal{T}_\alpha$(Goldberg [29], pp. 58): Let $\mathbb{X}$ be a Banach space and $\mathcal{T}_\alpha \in \mathcal{BL}(\mathbb{X})$, where α is a complex number. Again let $\mathcal{R}(\mathcal{T}_\alpha)$ and $\mathcal{T}_\alpha^{-1}$ denote the range and inverse of the operator $\mathcal{T}_\alpha$, respectively. Then the following possibilities may occur:
(A) $\mathcal{R}(\mathcal{T}_\alpha) = \mathbb{X}$,
(B) $\mathcal{R}(\mathcal{T}_\alpha) \neq \overline{\mathcal{R}(\mathcal{T}_\alpha)} = \mathbb{X}$,
(C) $\overline{\mathcal{R}(\mathcal{T}_\alpha)} \neq \mathbb{X}$,

and

(1) $\mathcal{T}_\alpha$ is injective and $\mathcal{T}_\alpha^{-1}$ is continuous,

(2) $\mathcal{T}_\alpha$ is injective and $\mathcal{T}_\alpha^{-1}$ is discontinuous,

(3) $\mathcal{T}_\alpha$ is not injective.

Remark 1.2. *Combining (A), (B), (C) and (1),(2), (3); we get nine different cases. These are labelled by $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3. The notation $\alpha \in A_2\sigma(\mathcal{T}, \mathbb{X})$ means the operator $\mathcal{T}_\alpha \in A_2$, i.e., $\mathcal{R}(\mathcal{T}_\alpha) = \mathbb{X}$ and $\mathcal{T}_\alpha$ is injective but $\mathcal{T}_\alpha^{-1}$ is discontinuous. Similarly others.*

Remark 1.3. *If α is a complex number such that $\mathcal{T}_\alpha \in A_1$ or $\mathcal{T}_\alpha \in B_1$, then α belongs to the resolvent set $\rho(\mathcal{T}, \mathbb{X})$ of $\mathcal{T}$ on $\mathbb{X}$. The other classification gives rise to the fine spectrum of $\mathcal{T}$.*

Definition 1.3. (Maddox [43], pp. 220-221) *Let λ, μ be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0$. For every $x = (x_k) \in \lambda$ and every integer n, we write*

$$A_n(x) = \sum_k a_{nk} x_k,$$

where the sum without limits is always taken from $k = 0$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (\lambda, \mu)$ if and only if $Ax \in \mu$ whenever $x \in \lambda$.

Lemma 1.10. (Wilansky [55], pp. 129) *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $\mathcal{T} \in \mathcal{BL}(c_0)$ from c_0 to itself if and only if*
(1) the rows of A in l_1 and their l_1 norms are bounded,
(2) the columns of A are in c_0.

Note: The operator norm of $\mathcal{T}$ is the supremum of the l_1 norms of the rows.

Lemma 1.11. (Wilansky [55], pp. 126) *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $\mathcal{T} \in \mathcal{BL}(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Note: The operator norm of $\mathcal{T}$ is the supremum of the l_1 norms of the columns.

Lemma 1.12. (Goldberg [29], pp. 59) *$\mathcal{T}$ has a dense range if and only if $\mathcal{T}^*$ is one to one, where $\mathcal{T}^*$ denotes the adjoint operator of the operator $\mathcal{T}$.*

Lemma 1.13. (Goldberg [29], pp. 60) *The adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is onto if and only if $\mathcal{T}$ has a bounded inverse.*

1.3 Outline of the thesis

The work presented in this thesis has two parts, part-A and part-B. The part-A contains Chapter-2 to 5 and part-B have chapter-6 and 7. Part-A is concerned with approximations of eigenvalue problem for a compact integral operator whereas in part-B is all about fine spectrum of generalized second order forward difference operator on sequence space c_0 and l_1.

In chapter-2, we consider the approximation of eigenelements of a compact integral operator with a smooth kernel by the Galerkin method using wavelet bases. By truncating the Galerkin operator, we obtain a sparse representation of a matrix eigenvalue problem. We obtain error bounds for the eigenelements. We show that the truncated wavelet Galerkin method is computationally economic in comparison to the Galerkin method using spline bases. Using elementary and double iteration refinement schemes, we obtain the convergence rates for both eigenvalue and eigenvector iterates.

In chapter-3, we consider approximation of eigenelements of a two-dimensional compact integral operator with a smooth kernel by discrete Galerkin (discrete collocation) and iterated discrete Galerkin (iterated discrete collocation) methods. By choosing numerical quadrature rule appropriately, we obtain error bounds for eigenelements for both discrete Galerkin (discrete collocation) and iterated discrete Galerkin (iterated discrete collocation) methods. We propose an asymptotic error expansion of the iterated discrete Galerkin and iterated discrete collocation method. Then we obtain asymptotic error expansion of approximate eigenvalues. We then apply Richardson extrapolation to obtain improved error bounds for the eigenvalues.

In chapter-4, we consider Galerkin and collocation methods for eigenvalue problem of compact integral operator for smooth kernels using Legendre polynomials basis. Also we consider Legendre Galerkin method for eigenvalue problem of weakly singular kernel. We obtain similar superconvergence rates for the eigenelements using global polynomial bases

as in the case of piecewise polynomial bases both in L^2 and infinity norm. We also show that the derivatives of iterated eigenvector converges to the corresponding derivatives of exact eigenvector with same order of convergence.

In chapter-5, we consider M-Galerkin and M-collocation methods for eigenvalue problem of a compact integral operator with smooth kernel using Legendre polynomials basis. We obtain the improved superconvergence rates for the eigenelements in both L^2 and infinity norm. We also show that the derivatives of iterated eigenvector converges to the corresponding derivatives of exact eigenvector with same order of convergence.

Numerical examples are presented to illustrate theoretical estimate from Chapter-2 to Chapter-5.

In chapter-6 , we introduce the generalized second order difference operator Δ^2_{uvw} on sequence space c_0, where $u = (u_k)$, $v = (v_k)$ and $w = (w_k)$, $k \in \mathbb{N}_0$ are real sequences under some restrictions. We have obtained the results on spectrum and point spectrum for the operator Δ^2_{uvw} over sequence space c_0. We have also obtained the results on continuous spectrum, residual spectrum and fine spectrum of the operator Δ^2_{uvw} on sequence space c_0.

In chapter-7, we have obtained the results on spectrum and point spectrum for the operator Δ^2_{uvw} over the sequence space l_1. We have also obtained the results on continuous spectrum, residual spectrum and fine spectrum of the operator Δ^2_{uvw} on sequence space l_1.

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