

## *Abstract*

This thesis addresses the development of enumeration and solution strategies for combinatorial discrete structures, namely, latin squares, partial latin squares and Sudoku squares. The problems related to these structures come into the category of combinatorial optimization problems and are fascinating the scientific community from several disciplines such as Computer Science(CS), Operational Research (OR), Combinatorics and many other applied areas. The main objective of the thesis is to explore the problems related to these discrete structures further and develop efficient algorithms for solving those problems. Counting the total number of latin squares and Sudoku squares is a very interesting problem, but it takes a huge amount of time and effort to deal with it as the size of these structures increases. It necessitates the development of efficient algorithms for this problem. There are previous approaches which have given good methods to deal with this problem, but there was not any systematic algorithmic approach for it. In this work, an innovative algorithmic approach has been proposed to deal with the enumeration problem of latin squares. A relation between permutation matrices and latin squares has been used. It is known that these permutation matrices yield permutations. An algorithm is proposed for counting the total number of latin squares of rank  $n$ , where  $n$  is a positive integer. For this, a relation between latin squares of rank  $n$ , and the sets of  $n$  mutually disjoint permutations is used. The worst-case time complexity of the algorithm has been provided, and the correctness of the algorithm is established. The experimental results are carried out for latin squares up to rank 10 and are verified with the results of previous approaches. Along with the latin squares, this method is also used for Sudoku squares, where the concept of S-permutation matrices and S-permutations is used. The proposed algorithm enumerates and generates all the subsets of cardinality  $n^2$  of all S-permutations which are mutually disjoint to each other, thereby, giving Sudoku squares. It is an exponential-time algorithm and the correctness of the algorithm is established. A mathematical formula is also derived. It is shown that our algorithm is more systematic and better in terms of computational efficiency in comparison to the brute force approach. The proposed algorithm verifies the results obtained for generally used Sudoku squares of size  $n^2 = 9$ . Next, a graph-based approach is proposed for the enumeration of Sudoku squares. A graph whose vertices represents S-permutations is constructed, where two vertices are connected by an edge if and only if their corresponding S-permutations are not disjoint to each other. A pivot vertex is selected and an induced subgraph is derived by considering all mutually disjoint vertices to the pivot. Then, a relation between maximum independent sets of such a graph and a Sudoku square is established. Next, an algorithm is given to count all the maximum independent sets of the graph which give all Sudoku squares of rank  $n$ . The correctness

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of the algorithm is shown and the algorithm is experimentally tested for Sudoku squares of size up to 9. There are some previous and recent efforts for providing bounds on the number of these squares. In this work, a mathematical expression for the upper bound on the total number of Sudoku squares is also given. Next, a partial latin square extension problem is considered. This problem is concerned with extending a given partial latin square as much as possible in terms of the number of its filled cells. It becomes very challenging for higher-order partial latin squares. Because of its applicability in various real-life scenarios, the researchers have carried out some good works for this problem. Some approximation algorithms and heuristics have been given to solve this problem, but there was not any application of evolutionary algorithm for it. Thus, in order to examine the applicability of evolutionary algorithm for this problem, genetic algorithm has been used. Next, the problem of obtaining solutions for partial Sudoku squares is considered. An improved genetic algorithm is given that applies multiple inversion operator and a swapping operator to solve these puzzles. Now, a problem with genetic algorithm is that it might lead to premature convergence. There is another metaheuristic approach, namely, simulated annealing which has been quite successful for combinatorial optimization problem. A limitation of simulated annealing is that it works with a single solution at a time, so it does not provide an entire view of the search space. Thus, a new hybrid metaheuristic approach which uses the merits of simulated annealing and the merits of genetic algorithm has been developed to solve partial Sudoku squares. The proposed algorithm iteratively explores solution search space by the improved genetic algorithm which is proposed in this work and then exploits it using simulated annealing algorithm to converge to the optimal solution. All the approaches are validated by experimenting them on a number of problem instances of various difficulty levels. The number of generations and the amount of CPU time (in seconds) are used as the performance measures. The experimental results are summarized in tables and graphs and it is observed that our approaches produce promising and better results.

Finally, the conclusions are summarized, and the scope of future works for the extension of the work of this thesis is also provided.

**Keywords:** Latin squares, Partial latin square extension problem, Sudoku squares, S-permutation matrices, S-permutations, Metaheuristic, Genetic algorithm, Simulated annealing, Hybridization, .