CHAPTER 1

Introduction

1.1 Graphs

In this thesis by a *graph* we mean a simple graph $G = (V, E)$, where V is a finite nonempty set and E is a subset of the set of all 2-subsets of V. The elements in V and E are called the *vertices* and *edges* respectively. Sometimes the vertex and the edge sets of G are denoted by $V(G)$ and $E(G)$ respectively. The total number of vertices in $V(G)$ is called the *order* of G. Two vertices x and y are said to be *adjacent* if $\{x, y\}$ is an edge; otherwise they are non-adjacent. If $e = \{x, y\}$ is an edge then both x and y are called the end vertices of e. If two vertices are adjacent then each is called a *neighbour* of the other vertex. The complement of a graph G is the graph $\overline{G}$ on the same vertices of G such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

A graph G is called a *complete graph* if for any two vertices x and y in G, $\{x, y\}$ is an edge. In this thesis we denote a complete graph with n vertices by the symbol K_n. A graph G is called a *bipartite graph* if the vertex set $V(G)$ can be partitioned into two nonempty subsets X and Y in such a way that each edge of G has one end vertex in X and another end vertex in Y. The partition $\{X, Y\}$ is called a bipartition of G. A *complete bipartite* graph G is a bipartite graph with bipartition $V(G) = X \cup Y$, such that every vertex in X is adjacent to every vertex in Y. If $|X| = m$ and $|Y| = n$, then a complete bipartite graph is denoted by $K_{m,n}$. A graph G is called *t-partite graph* if the vertex set $V(G)$ can be partitioned into two nonempty subsets $X_1, X_2, \ldots, X_t$ such that each edge of G has one end vertex in X_i and the other end vertex in X_j, $1 \leq i \neq j \leq t$. A *complete t-partite graph* is a t-partite graph with partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_t$ such that every vertex in X_i is adjacent to every vertex in X_j, $1 \leq i \neq j \leq t$. If $|X_i| = n_i$, for $i = 1, 2, \ldots, t$, then a complete t-partite graph is denoted by $K_{n_1, n_2, \ldots, n_t}$.

A graph $H = (V_1, E_1)$ is said to be a *subgraph* of a graph $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$. Further H is called an *induced subgraph* if H contains all the edges of G whose end vertices are in H. The *degree* of a vertex x in a graph G is the number of vertices in G which are adjacent to x. In this thesis we denote the degree of a vertex x in a graph G by $deg_G(x)$ (or simply $deg(x)$, if the graph is understood). A degree one vertex is called a *pendant* vertex or a *leaf*.

A *walk* in a graph G is a finite sequence $W : v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ whose terms are alternately vertices and edges (starting and ending with vertices) such that, for $1 \leq i \leq k$, the edge e_i has end vertices v_{i-1} and v_i. If $v_0 = v_k$ then W is called a closed walk. If the vertices $v_0, v_1, \ldots, v_k$ of the walk $W : v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ are distinct then W is called a *path* (or a $v_0 - v_k$ *path*). A path on n vertices is denoted by P_n. The number of edges in a path is called its *length*. If G is a connected graph then the distance between two vertices u and v in G, denoted by $d_G(u, v)$ (or simply $d(u, v)$ if the graph is understood), is the minimum of the lengths of all $u - v$ paths in G. A graph G is said to be *connected* if each pair of vertices in G belongs to a path; otherwise G is called *disconnected*. The *eccentricity* of a vertex v in a connected graph G is denoted by $e(v)$ and defined as $e(v) = \max\{d(u,v) : u \in V(G)\}$. The diameter of a connected graph G, denoted by $diam(G)$, is given by $\max\{e(v) : v \in V(G)\}$. A *cycle* is a closed walk $W : v_0e_1v_1e_2v_2 \ldots v_{k-1}e_kv_0$ where all the vertices are distinct. A cycle with n vertices is denoted by C_n. A *tree* is a connected graph without any cycle. A *star* is a tree having a vertex which is adjacent to all the other vertices. A star with n vertices is the bipartite graph $K_{1,n-1}$. A *rooted tree* is a tree with one vertex chosen as the root. A complete m-ary tree is a rooted tree such that the degree of the root vertex is m, the degree of all other non-pendant vertices is equal to $m+1$, and all pendant vertices are of the same distance from the root. A tree T is said to be a *m-distant* tree if there is a path P of maximum length in T such that every vertex in T is of distance at the most m from P. This path P is called a *central path* of T. Every tree is an m-distant tree for some m. An 1-distant tree is called a *caterpillar* and a 2-distant tree is called a *lobster*. The n-dimensional *cube* or *hypercube* Q_n is the graph whose vertices are the n-tuples with entries in $\{0,1\}$ and its edges are the pairs of n-tuples that differ in exactly one position. The cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is the graph whose vertex set is the cartesian product $V(G_1) \times V(G_2)$, and any two vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1. One can easily verify that $G_1 \square G_2 = G_2 \square G_1$. The hypercube Q_n can be expressed as the cartesian product of n copies of K_2. The *book graph* B_m is defined as the cartesian product $S_{m+1} \square P_2$, where S_{m+1} is the star graph $K_{1,m}$. The cartesian product $S_{m+1} \square P_n$ is called the *stacked book graph* $B_{m,n}$. The *prism* graph Y_m is the cartesian product $C_m \square P_2$. A two dimensional grid graph is the cartesian product $P_m \square P_n$. For any graph G and a positive integer r, the r^{th} power of G, denoted by G^r, is the graph with the vertex set same as that of G and two vertices u and v are adjacent in G^r if and only if $d_G(u,v) \le r$. For any list l chosen from $\left\{1,2,3,\dots,\left\lfloor \frac{n}{2} \right\rfloor\right\}$, a *circulant graph* $Ci_n(l)$ is a graph on the vertices $v_0, v_1, \dots, v_{n-1}$ such that each v_i, $0 \le i \le n-1$, is adjacent to v_{i+j} and v_{i-j} (subscripts are taken *mod* n) for every j in the list l. The circulant graph $Ci_n\left(1,2,\dots,\left\lfloor \frac{n}{2} \right\rfloor\right)$ is the complete graph K_n and the graph $Ci_n(1)$ is the cycle C_n. The *generalized Petersen graph* $GP(n,r)$, for $n \ge 3$ and $1 \le r \le \left\lfloor \frac{n-1}{2} \right\rfloor$, is the graph with vertex set $\{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+r}\}, i = 0,1,2,\dots,n-1\}$, where subscripts are to be read modulo n.

1.2 Motivation for Radio k-colorings of Graphs

The Frequency Assignment Problem (FAP) is the problem of assigning frequencies to transmitters in some optimal manner that avoids interferences. Nearly three decade back this problem has been modeled as a graph coloring (labeling) problem see, Hale (1980). This coloring has several variations depending upon the type of assignment of frequency to transmitters. In this section, we discuss some of them including radio k-coloring of graphs.

The FAP plays an important role in wireless network and is a well-studied interesting problem. Due to rapid growth of wireless networks and to the relatively scarce radio spectrum the importance of FAP is growing significantly. Many researchers have modeled FAP as an optimization problem as follows: Given a collection of transmitters to be assigned operating frequencies and a set of interference constraints on transmitter pairs, find an assignment that satisfies all the interference constraints and minimizes the value of a given objective function. Hale (1980) has modeled FAP as both frequency-distance constrained and frequency constrained optimization problems. Since then, a number of graph colorings have been inspired by the FAP.

T-colorings of graphs is also a model of FAP, studied by Liu (1992, 1994, 1996, 2000), Cozzens et al. (1982, 1991), Griggs et al. (1994), Rabinowitz et al. (1985), Raychaudhuri (1994) and Tesman (1989). For any simple connected graph G and a list T of non-negative integers containing 0, T-coloring is a mapping f from the vertex set V of G to the set of non-negative integers $\{0, 1, 2, \dots\}$ such that $|f(x) - f(y)| \notin T$ whenever x and y are adjacent in G. The span of a T-coloring f is the difference between the largest and the smallest numbers in $f(V)$, i.e., $max\{|f(x) - f(y)| : x, y \in V\}$. The minimum span among all T-colorings of G is called T-span of G.

Roberts (1988) proposed an FAP with two levels of interference which Griggs (1992) adapted to graphs and extended to a more general graph problem of distance-constrained coloring as follows. For nonnegative integers $p_1, \dots, p_m$, an $L(p_1, \dots, p_m)$-coloring of a graph G is a coloring of its vertices by nonnegative integers such that vertices at distance exactly i receive labels that differ by at least p_i. The maximum label assigned to any vertex is called the *span* of the coloring. The goal of the problem is to construct an $L(p_1, \dots, p_m)$-coloring of the smallest span. Sometime the distance constraints are considered to decrease with the distance which were studied by Bertossi et al. (2003), i.e., $p_1 \geq p_2 \geq \dots \geq p_m \geq 1$. However, there are also practical applications of the case $p_1 = 0$, $p_2 = 1$ studied by Bertossi et al. (1995) and Jin et al. (2005). Distance-constrained coloring problem is related to ordinary graph coloring problem of graphs. If $p_1 = \dots = p_m = 1$, then the problem reduces to the coloring of the k^{th} power of the graph G. So many results on colorings of graph powers translate to distance-constrained coloring and vice versa (for example see Molloy et al. (2005, 2002)).

More popular distance-constrained coloring is the $L(2, 1)$-coloring of graphs. A major open problem in this coloring is the conjecture of Griggs and Yeh (1992) that asserts that every graph G with maximum degree $\Delta \geq 2$ has an $L(2, 1)$-

coloring of span at most Δ^2. Lots of research work has been done on $L(2,1)$-coloring of graphs and their algorithmic aspects (for example see Bella et al. (2007), Calamoneri (2006), Chang et al. (2000b, 1996), Georges et al. (1995, 2003), Král' (2004) and Mc Diarmid (2003)). An another version of FAP is described by a graph G in which each edge e is assigned a positive integer weight $w(e)$ and the problem is to give colorings to the vertices of G with positive integers such that the colors of adjacent vertices u and v differ by at least $w(uv)$. A survey on this problem is made by Mc Diarmid (2001). Instead of integer coloring in the above FAP real number colorings are also considered by some researchers, a survey of which given by Griggs et al. (2009).

Radio k-colorings of graphs is also a type of FAP . In fact a radio k-coloring is the same as $L(k, k-1, \ldots, 2, 1)$ coloring, in which the colors are the positive integers. This problem is also a generalization of the usual vertex coloring problem. In a proper vertex coloring of a graph two vertices at distance one get different colors whereas in radio k-coloring two vertices having distance less than or equal to k have to get different colors satisfying a condition. In this thesis, we are interested on radio k-colorings of graphs, $1 \le k \le diam(G)$, suggested by Chartrand et.al (2001). The term "radio k-coloring" emanates from its connection with the Frequency Assignment Problem.

1.2.1 Graph Theoretic Modeling

Suppose that nine radio stations $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$ and s_9 are planning to start broadcasting in a country. We want to assign a frequency to each station so that no two stations interfere with each other. Depending on interference area of stations and their geometric locations we draw circles (of the same diameters) taking these stations as centers as shown in Figure 1.1. The diameter of circles should be less than or equal to the diameters of the interference area of stations to ensure the interference between stations whose respective circles intersect. If a station is such that it is not coming within the interference area of any other station, then we can leave that station as we can assign any frequency to it. Now we construct a graph taking radio stations as vertices and two vertices are adjacent if the corresponding circles intersect. Then we will get a connected graph as shown in Figure 1.2.

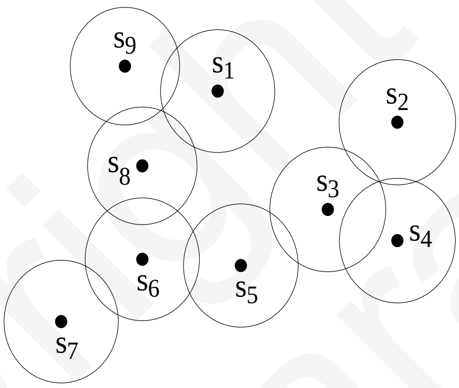

Figure 1.1: Interference structure of radio stations.

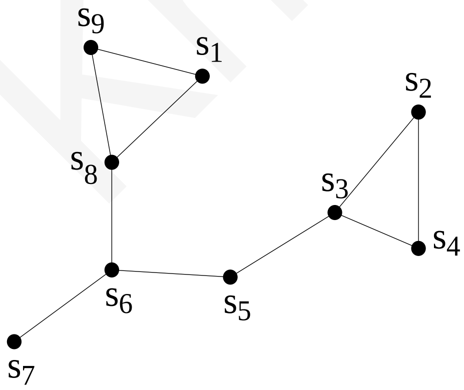

Figure 1.2: Graph theoretical model of frequency assignment problem.

Generally diameters of the interference area of stations will be less than or equal to the diameter of the graph except the case when each station is in the interference area of all other stations. In that case we will get a complete graph and it is easy to assign frequencies without interference. We choose a positive integer k such that beyond the graph theoretical distance k there will not be any interference. So we can assign same frequency to two vertices of $k+1$ distance apart.

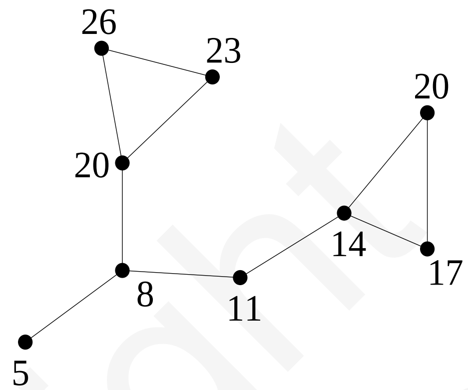

Figure 1.3: Assignment of frequencies with $k = 3$ and minimum frequency 5.

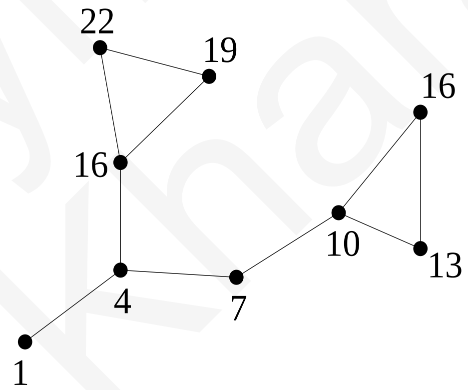

Figure 1.4: Assignment of frequencies with $k = 3$ and with minimum frequency 1.

Figures 1.3, 1.4 and 1.5 are all assignments of frequencies when $k = 3$. Observe that the assignment shown in Figure 1.4 is obtained by subtracting 4 from each frequency in the assignment shown in Figure 1.3. So we can say that in our assignment we should use 1 (minimum available frequency) to some vertex as our aim is to reduce the maximum frequency assigned. The maximum frequency assigned in Figure 1.4 is still reduced in Figure 1.5. So the main aim here is that

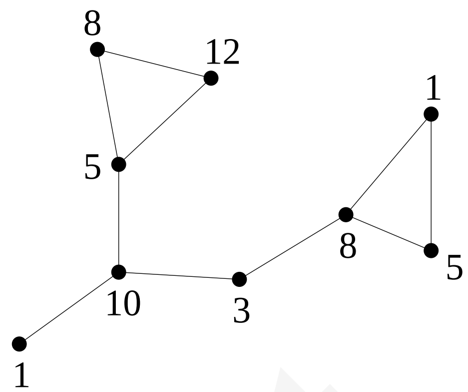

Figure 1.5: Assignment of frequencies with $k = 3$ and reduced maximum frequency.

minimizing the highest frequency assigned and this is what the *radio k-coloring* problem (precise mathematical definition will be given in the following subsection).

1.2.2 Terminology and Definitions

Definition 1.1 Let G be a simple connected graph with diameter d and k, $1 \leq k \leq d$, be an integer. A *radio k-coloring* of G is an assignment f of positive integers to the vertices of G such that for any two distinct vertices u and v,

$$|f(u) - f(v)| \geq 1 + k - d(u, v), \tag{1.1}$$

where $d(u, v)$ is the distance between u and v in G.

In this thesis we refer (1.1) as the *radio condition.*

Definition 1.2 Let f be a radio k-coloring of a simple connected graph G. The *span* of f, $rc_k(f)$, is $\max\{f(u)\colon\ u \in V(G)\}$. The *radio k-chromatic number*, $rc_k(G)$, of G is min{ span of f: f is a radio k-coloring of G}. A radio k-coloring of G having the span $rc_k(G)$ is called a *minimal radio k-coloring.*

Example 1.1 In the following figures, we give two different radio k-colorings of the same graph.

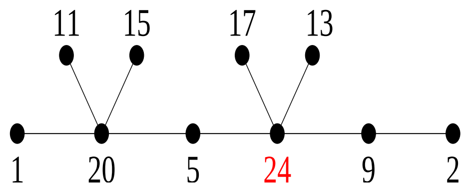

Figure 1.6: Radio 5-coloring with span 24.

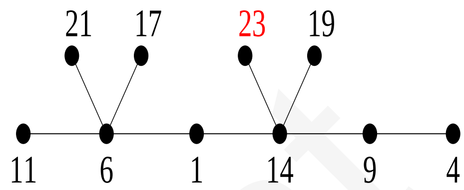

Figure 1.7: Radio 5-coloring with span 23.

It is not difficult to give a radio k-coloring to any graph. But finding radio k-chromatic number is challenging.

In the literature, for some special values of k there are special names of radio k-coloring and radio k-chromatic number which are given below.

k	*Name of coloring*	$rc_k(G)$
1	Usual coloring	Chromatic number, $\chi(G)$
$diam(G)$	Radio coloring	Radio number, $rn(G)$
$diam(G) - 1$	Antipodal coloring	Antipodal number, $ac(G)$
$diam(G) - 2$	Nearly antipodal coloring	Nearly antipodal number, $ac'(G)$

1.3 Literature Survey

Radio k-coloring was introduced by Chartrand et al. (2001). Two survey papers are published on radio k-colorings, one is by Chartrand et al. (2007) and the other is by Panigrahi (2009). Since finding radio k-chromatic number is highly nontrivial, it is known for very few graphs and lower and upper bounds have been given for some graphs. Only radio numbers of graphs are studied by several authors and so more results are there on this than the radio k-chromatic number of graphs for k different from the diameter. Therefore we divide the survey part into two different subsections.

1.3.1 Known Results on Radio Number of Graphs

In their introductory paper Chartrand et al. (2001) have studied radio number of some well known graphs, namely, cycles, complete multipartite graphs, and graphs with diameter 2. They have shown that $rn(G) = n + t - 1$ if and only if the minimum number of components in a spanning linear forest of the complement $\overline{G}$ of G is t, where a linear forest is a graph all of whose components are paths. They have computed the radio numbers of C_n, for $n \leq 8$, and have given bounds for other values of n. For $n \geq 9$, the lower bound of $rn(C_n)$ is $3\lceil \frac{n}{2} - 1 \rceil$ and the upper bound is $\frac{(n-1)}{2}^2 + 1$ for n odd and $\frac{n}{2}^2 - \frac{n}{2} + 2$ for n even. The radio number of a complete t-partite graph $K_{n_1,\ldots,n_t}$ is $\sum_1^t n_i + (t - 1)$. Bounds for $rn(G)$ of a connected graph G of order n and diameter d were given by Chartrand et al. (2005) as $rn(P_{d+1}) \leq rn(G) \leq rn(P_{d+1}) + (n - d - 1)d$. In the same paper they have presented two lower bounds for $rn(G)$, first in terms of the diameter d and the maximum degree Δ of G, and the second in terms of the diameter and the clique number ω of G. They are $2 + \Delta(d - 1)$ and $1 + d(\omega - 1)$ respectively.

For paths and cycles, the radio numbers were studied by Chartrand et al. (2005) and Zhang (2002), while the exact value remained open until solved by Liu and Zhu (2005). They have determined the exact value of radio number of path P_n, $n \geq 4$, as $2p^2 - 2p + 2$ if $n = 2p$ and $2p^2 + 3$ if $n = 2p + 1$. They have also determined the exact value of radio number of C_n, $n \geq 3$, as $\frac{n-2}{2}\phi(n) + 2$ if $n \equiv 0, 2\ (mod\ 4)$ and $\frac{n-1}{2}\phi(n) + 1$ if $n \equiv 1, 3\ (mod\ 4)$, where $\phi(n)$ is equal to $s + 1$ if $n = 4s + 1$ and is $s + 2$ if $n = 4s + r$ for some $r = 0, 2$ or 3.

Fernandez et al. (preprint) have given the radio number of the complete graph K_n and the wheel graph W_n as n and $n + 2$ respectively, where the wheel graph

W_n consists of a cycle C_n together with a center vertex that is adjacent to all vertices of the C_n. They have also computed $rn(G_n) = 4n + 2$, $n \geq 4$, where G_n is the n-gear graph obtained from W_n by inserting one vertex between each pair of vertices on the main cycle C_n of W_n.

The radio number $rn(C_n^2)$, n even, of the square of even cycles are completely determined by Liu and Xie (2004). They are $\frac{2p^2-5p-1}{2}$ if $n = 4p$ and p is odd, $\frac{2p^2+3p}{2}$ if $n = 4p$ and p is even, $p^2 + 5p + 1$ if $n = 4p + 2$ and p is odd, and $p^2 + 4p + 1$ if $n = 4p + 2$ and p is even. In case of square of odd cycle C_n^2, $n = 4p + 1$, they have found out that the value of $rn(C_n^2)$ as $p^2 + 2p$ if p is even and $p^2 + p$ if $p \equiv 3 \pmod 4$. Moreover, for the case $p \equiv 1 (\text{mod } 4)$, they have given bounds as $p^2 + p + 1 \leq rn(C_{4p+1}^2) \leq p^2 + p + 2$. For $n = 4p + 3$, the value of $rn(C_n^2)$ is given as follows: for some positive integer m, if $p = 4m$, or $p = 4m + 2$ for $m \not\equiv 5(mod\ 7)$, then $rn(C_n^2)$ is $\frac{2p^2+9p+6}{2}$; $\frac{2p^2+9p+6}{2} \leq rn(C_n^2) \leq \frac{2p^2+9p+12}{2}$ if $p = 4m + 2$, $m \equiv 5(mod\ 7)$; $\frac{2p^2+7p+5}{2}$ if $p = 4m + 1$, $m \equiv 0, 1(mod\ 3)$; $\frac{2p^2+2p+7}{2}$ if $p = 4m + 1$, $m \equiv 2(mod\ 3)$; $\frac{2p^2+7p+7}{2}$ if $p = 4m + 3$, $m \equiv 0(mod\ 3)$; and $\frac{2p^2+7p+5}{2}$ if $p = 4m + 2$, $m \equiv 1, 2(mod\ 3)$. Liu and Xie (2009) have determined the exact value of radio number of the square path P_n^2, $n \geq 9$, as $p^2 + 2$ if $n \equiv 1(mod\ 4)$, and $p^2 + 1$ otherwise, where $p = \lfloor \frac{n}{2} \rfloor$. Sooryanarayana and Raghunath (2007) have determined the radio number of the cube of a cycle, C_n^3, for all $n \leq 20$ and for $n \equiv 0$ or 2 or 4 (mod 6).

Next, Liu (2008) has studied the radio number of trees. She has given a lower bound for the radio number $rn(T)$ of an n-vertex tree with diameter d as $(n - 1)(d + 1) + 1 - 2w(T)$, where $w(T)$ is the weight of T defined as $w(T) = \min_{u \in V(T)} \left\{ \sum_{v \in V(T)} d(u, v) \right\}$. She also has characterized the trees achieving this bound. A spider is a tree with at most one vertex of degree more than two. If the spider has no vertex with degree more than two then it is a path and the radio number has already been discussed for this. If the spider has a vertex v of degree more than two, say m, then the spider will have m number of paths with one end at v

and the other at a pendant vertex. If the length of these paths be $l_1, l_2, l_3, \ldots, l_m$ with $l_1 \geq l_2 \geq \cdots \geq l_m$, $m \geq 3$, then the spider is denoted by $S_{l_1,l_2,l_3,\ldots,l_m}$. Liu (2008) has also given a lower bound for the radio number of $S_{l_1,l_2,l_3,\ldots,l_m}$ as $\sum_{i=1}^{m} l_i(l_1 + l_2 - l_i) + \left\lceil \frac{l_1 - l_2}{2} \right\rceil \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor + 1$ and has characterized the spiders achieving this bound.

Li et al. (2010) have determined the radio number of complete m-ary trees ($m \geq 3$) with height k (≥ 2), denoted by $T_{k,m}$, as $\frac{m^{k+2}+m^{k+1}-2km^2+(2k-3)m+1}{(m-1)^2}$.

1.3.2 Known Results on Other $rc_k(G)$

For general k, $1 \leq k \leq d$, radio k-chromatic number of very few graphs are known so far, even for all paths these numbers have not yet been determined completely.

Upper and lower bounds were established for $ac(G)$ of a graph G with diameter $d \geq 3$, and n vertices, by Chartrand et al. (2002), as $1+\frac{d^2}{4} \leq ac(G) \leq \frac{(d-1)(2n-d-2)}{2}+1$. Also, they have determined two lower bounds for the same first in terms of the diameter d and the maximum degree Δ and the second in terms of the diameter d and the clique number ω as $2+\Delta(d-2)$ and $1+(d-1)(\omega-1)$ respectively. For $n > k$ Chartrand et al. (2004) have given an upper bound for the radio k-chromatic number, $rc_k(P_n)$, of path P_n as $\frac{k^2+2k+1}{2}$ if k is odd and $\frac{k^2+2k+2}{2}$ if k is even. Also they have given a lower bound for the same as $\frac{k^2+3}{4}$ if k is odd and $\frac{k^2+4}{4}$ if k is even. Panigrahi and Kola (2009) have improved the upper bound of $rc_k(P_n)$ for some values of k and n. Kola and Panigrahi (2010b, communicated) have further improved the upper bound of $rc_k(P_n)$ for $n \leq \left\lfloor \frac{k^2+2k}{2} \right\rfloor$.

Although Chartrand et al. (2005) have defined radio k-colorings of graphs G for $1 \leq k \leq diam(G)$, from the mathematical point of view one can also see this problem for $k > diam(G)$ as this may be useful to find radio k-chromatic number of supergraphs H of G with bigger diameter than that of G. Therefore Kchikech et al. (2007) have given exact value of $rc_k(P_n)$ for $k \geq n$ as $(n-1)k - \frac{1}{2}n(n-2) + 1$

if n is even and $(n-1)k - \frac{1}{2}n(n-1)^2 + 2$ if n is odd. Also they have improved the lower bound for the same when $1 \le k \le n-1$ as $\frac{k^2+3}{2}$ if k is odd and $\frac{k^2+6}{2}$ if k is even. Panigrahi and Kola (2009) have improved the above lower bound of $rc_k(P_n)$, k odd, from $\frac{k^2+3}{2}$ to $\frac{k^2+5}{2}$. Liu and Zhu (2005) determined the exact value of radio $(n-1)$-chromatic number (which is the radio number) of path P_n (that has already been discussed in the previous subsection). Khennoufa and Togni (2005) determined the exact value of the radio $(n-2)$-chromatic number (i.e., radio antipodal number $ac(P_n)$) of the path P_n as $2p^2 - 4p + 5$ if $n = 2p$ and $2p^2 - 2p + 3$ if $n = 2p+1$. Although these numbers are correct, Kola and Panigrahi (2009a) found an error in their proof of the lower bound and have established the same in a different way.

Let d be the diameter of the cycle C_n on n vertices. Juan and Liu (2006) have determined $rc_{d-1}(C_n)$ as $2s^2+1$ when $n = 4s+1$, $2s^2+2s+1$ when $n = 4s+3$ and have given an upper and lower bounds when $n = 4s$ as $2s^2$ and $2s^2 - \lfloor \frac{s}{2} \rfloor$ respectively. Whereas Chartrand et al. (2000) have computed the exact value of the same when $n = 4s+2$. See p. 27.

Kchikech et al. (2008) have given upper and lower bounds for the radio k-chromatic number, $rc_k(G \Box G')$, of the cartesian product $G \Box G'$ of any two graphs G and G' of order n and m respectively, as below. The upper bound is $m(rc_k(G) + (m-1)k - \sum G' + 1)$ for $k \ge diam(G \Box G') - 1$ and the lower bound is $(mn-1)(k+1) - m\sum_c G - n\sum_c G' + 2$ for $k \ge 1$, where for any graph H with the vertex set $V(H) = \{v_0, \ldots v_{p-1}\}$, $\sum H$ and $\sum_c H$ are defined as $\sum H = max_\pi \sum\limits_{i=0}^{p-2} d(\pi(v_{i+1}), \pi(v_i))$, and $\sum_c H = max_\pi \sum\limits_{i=0}^{p-1} d(\pi(v_{i+1}), \pi(v_i))$ respectively, π is a permutation of $V(H)$ and indices are taken modulo p. Also they have given upper and lower bounds for radio k-chromatic number, $rc_k(Q_n)$, of hypercube Q_n as $(2^n - 1)k - 2^{n-1} + 1$ if $n - 1 \le k < 2n - 2$ and $(2^n - 1)k - 2^{n-1}(2n-3) - n$ if $k \ge 1$.

Kchikech et al. (2007) have found the exact value of the radio k-chromatic number of stars $K_{1,n}$ as $n(k-1)+2$ and have also given an upper bound for radio k-chromatic number, $rc_k(T)$, $k \geq 2$, of an arbitrary non-star tree T on n vertices as $(n-1)(k-1)$.

1.4 Basic Results

Recall that giving a radio k-coloring f to a graph is not at all difficult, but minimizing the span of f is the most difficult problem. It is clear that if for a radio k-coloring f and any pair of vertices u and v, $|f(u)-f(v)|$ is a large number then the span of f is also a large number. So naturally it comes that we should consider those radio k-colorings for which $|f(u)-f(v)|$ is as small as possible for every pair of vertices u and v. The lower bound of $|f(u)-f(v)|$ is $k+1-d(u,v)$. So we try to get those radio k-colorings for which $|f(u)-f(v)| = k+1-d(u,v)$, for every pair of vertices u and v. However, such radio k-colorings may not be possible for all graphs and so we go for minimizing the difference $|f(u)-f(v)|-(k+1-d(u,v))$ and arrive at the following definition.

Definition 1.3 For any radio k-coloring f of a simple connected graph G on n vertices and an ordering $x_1, x_2, \ldots, x_n$ of vertices of G with $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$, we define ϵ_i(or ϵ_i^f to specify the coloring f) $= (f(x_i)-f(x_{i-1}))-(1+k-d(x_i,x_{i-1}))$, $2 \leq i \leq n$. It is clear from definition of radio k-coloring that $\epsilon_i \geq 0$, for all i.

The above definition was given by Khennoufa et al. (2005) while finding the antipodal number of P_n.

Notation 1.1 Notice that the ordering of vertices in Definition 1.3 will be different for different radio k-colorings. However for the sake of convenience we use the same names $x_1, x_2, \ldots, x_n$ (in order) for the ordering of vertices (with respect to non-decreasing vertex coloring) corresponding to any radio k-coloring of the graph. On the other hand for the sum of distances between consecutive pairs in the ordering $x_1, x_2, \ldots, x_n$, we record the corresponding radio k-coloring i.e., by the notation $\sum_f d(x_i, x_{i-1})$ we mean the sum $\sum_{i=2}^{n} d(x_i, x_{i-1})$ where $x_1, x_2, \ldots, x_n$ is an ordering of the vertices according to non-decreasing order of vertex coloring by the radio k-coloring f.

The next theorem is important for most of the chapters in the thesis as we apply this to get minimal radio k-colorings.

Theorem 1.1 Let f be a radio k-coloring of G on n vertices.

(a) If $\max_g \sum_g d(x_i, x_{i-1}) = \sum_f d(x_i, x_{i-1})$ and $\min_g \sum_{i=2}^{n} \epsilon_i^g = \sum_{i=2}^{n} \epsilon_i^f$, where maximum and minimum is over all possible radio k-colorings g of G, then f is a minimal radio k-coloring of G.

(b) If $\max_g \sum_g d(x_i, x_{i-1}) = \sum_f d(x_i, x_{i-1})$ and $\sum_{i=2}^{n} \epsilon_i^f = 0$, then f is a minimal radio k-coloring of G.

Proof: Let g be an arbitrary radio k-coloring of G. Then

$$\begin{aligned} g(x_n) - g(x_1) &= \sum_{i=2}^{n} [g(x_i) - g(x_{i-1})] \\ &= \sum_{i=2}^{n} [1 + k - d(x_i, x_{i-1}) + \epsilon_i^g] \\ &= (n-1)(1+k) - \sum_g d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^g. \end{aligned}$$

Since $g(x_1) = 1$, we get,

$$g(x_n) = (n-1)(1+k) - \sum_g d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^g + 1 \tag{1.2}$$

If f satisfies the conditions in (a) part (i.e., f is a single radio k-coloring for which the distance sum is maximum and the ϵ-sum is minimum), then $rc_k(G) =$

$\min_g g(x_n) = (n-1)(1+k) - \max_g \sum_g d(x_i, x_{i-1}) + \min_g \sum_{i=2}^{n} \epsilon_i^g + 1 = (n-1)(1+k) - \sum_f d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1 = f(x_n) = rc_k(f)$. So f is a minimal radio k-coloring and we prove (a) part. Part (b) follows from (a) immediately because the smallest possible value of $\sum_{i=2}^{n} \epsilon_i^f$ is zero. □

Although the next lemma looks simple, it helps to get many important consequences that is seen in many results in the thesis.

Lemma 1.1 If H is a subgraph of a graph G, then $rc_k(H) \leq rc_k(G)$.

Proof: Every radio k-coloring f of G induces a radio k-coloring of H. So we get $rc_k(H) \leq \min_f rc_k(f) = rc_k(G)$, where minimum is taken over all possible radio k-colorings f of G. □

1.5 Chapterization

The thesis consists of seven chapters of which **Chapter 1** contains introduction, a brief survey of research carried out on radio k-colorings of graphs and basic results which will be used in the consequent chapters.

In **Chapter 2**, we give a lower bound for radio k-chromatic number of an arbitrary graph G in terms of k, number of vertices n and a positive integer M, where M is the smallest integer such that $d(u,v) + d(v,w) + d(w,u) \leq M$ for every triplet u, v and w in $V(G)$. We determine the M-value for cartesian product of two graphs G_1 and G_2 in terms of the M-values of G_1 and G_2, and the M-value of power of a graph G in terms of the M-value of G. Also, we give lower bounds to $rc_k(C_n)$, $rc_k(P_m \square P_n)$, $rc_k(C_m \square P_n)$ and $rc_k(S_{m+1} \square P_n)$. We see that the lower bound of $rc_k(C_n)$ obtained here coincides with the radio and antipodal numbers of C_n (which were determined earlier), when $k = diam(G)$ and $k = diam(G) - 1$.

As the hypercube Q_n plays an important role in networking, in **Chapter 3** we study radio k-coloring of this graph. We use the technique discussed in Chapter 2 and find a lower bound for $rc_k(Q_n)$. Further, we verify that this lower bound achieves the radio number of Q_n by giving a minimal radio coloring. Also, we determine the M-value of r^{th} power of Q_n, denoted by Q_n^r, from which we get a lower bound for $rc_k(Q_n^r)$.

We devote **Chapter 4** to study radio k-chromatic number of path P_n. We determine nearly antipodal number of P_n and found radio $(n-4)$-chromatic number of the same when n is odd and give an upper bound when n is even. Also, we improve the lower bound of $rc_k(P_n)$ for any odd integer $k \geq 3$. Kchikech et al. (2007) have conjectured that for a fixed k, $\lim_{n\to\infty} rc_k(P_n) = l$, where $l = \frac{k^2+2k+1}{2}$ if k is odd and $l = \frac{k^2+2k+2}{2}$ if k is even. We improve the existing upper bound of $rc_k(P_n)$, for any k and $n < \left\lfloor \frac{k^2+2k+2}{2} \right\rfloor$. Further, we conjecture that $rc_k(P_n) = \frac{k^2+2k+1}{2}$ when k is odd and $n \geq \frac{k^2+2k+1}{2}$ and $rc_k(P_n) = \frac{k^2+2k+2}{2}$ when k is even and $n \geq \frac{k^2+2k+2}{2}$. In this chapter, we also relate this conjecture with the conjecture on the antipodal chromatic number of C_n $(n \equiv 0 (mod\ 4))$.

In **Chapter 5**, we give some results on radio colorings of m-distant trees. We determine radio numbers of some classes of caterpillars and extend the technique to find radio numbers of some m-distant trees.

In **Chapter 6**, we give a way of getting minimal radio colorings of graphs. Also, we use this technique to define minimal radio colorings of some classes of generalized petersen graphs and circulant graphs.

Lastly, **Chapter 7** contains the conclusion and discussions on future scope of research.