CHAPTER 1

GENERAL INTRODUCTION

The investigations in connection with existence, nature and computation of the set of points x in the domain X of a self-mapping T such that Tx = x are the subject matter of fixed point theory and those points x are called the fixed points of T. One carries out such investigations by imposing various types of conditions on T and on X as well. The present thesis is just an endeavour in this direction.

The fixed point theory as it is today has been developed centering mainly around two earlier theorems, one of which is due to Banach (1922) and the other due to Schauder (1930). Because of the manifestation of their importance in application to diverse disciplines, many people looked at these theorems from various angles and thereby derived generalizations. However, Banach's theorem has been favoured against the other for its analytical simplicity and wider applicability. The thesis deals with nonexpansive mappings which are more general than Banach's contraction mappings. The results established here in connection with nonexpansive mappings are related to Schauder's and its extensions. Our main concern is to study the families of nonexpansive mappings and their common fixed points. Attention has been focussed on the existence and computation of fixed points and very little has been done to study the properties of fixed point sets.

The thesis has been divided into six chapters including the present one.

Chapter 2 is devoted to the discussion of Schauder fixed point theorems for nonexpansive and relatively nonexpansive mappings. Schauder theorem establishes the existence of a fixed point of a continuous mapping which leaves a compact and convex subset of a Banach space invariant. But the proof of the theorem is rather sophisticated. However, it has been shown by Dotson and Mann (1977) that the proof is an easy affair if the mappings happen to be nonexpansive. In this chapter the result of Dotson and Mann has been extended to a pair of mappings where one is nonexpansive relative to the other in the spirit of Jungck (1976). In the process the existence of a common fixed point of two mappings has been established. The result of Dotson and Mann is also valid if the invariant subset is star-shaped instead of being compact, see Dotson (1972a). The same has been observed to be true in our case also. Further, it has been illustrated with the help of an example that our result is indeed an extension.

Chapter 3 deals with composite nonexpansive mappings. Sometimes it may happen that a mapping f, though leaves a compact and convex subset of a Banach space invariant, is not nonexpansive but its composite fg or gf with another mapping g is nonexpansive. Then it is possible to discuss the existence of a fixed point of the composite in the framework of Chapter 2. This has been done here and, in the process, we have established the existence of a common fixed point of the composite and its constituents. The case of relative nonexpansive mappings has also been included in the discussion and an example has been provided for illustration.

Schauder theorem was extended to a family of affine, continuous and commuting mappings by Markov (1936) and Kakutani (1938). However, they imposed the condition of affineness on the mappings which was not there is Schauder's. DeMarr (1963) removed the condition of affineness, but imposed the condition of nonexpansiveness on the mappings. Later on Browder (1965) and Göhde (1965) improved DeMarr's result by removing the condition of compactness from the invariant subset. In fact, they took the invariant subset to be closed, bounded and convex in a uniformly convex Banach space. Kirk (1965) obtained the same result by assuming the Banach space to be reflexive with normal structure, the other conditions remaining the same.

Chapter 4 deals mainly with simple extensions of the results of Markov-Kakutani and Browder-Göhde, where the condition of convexity has been removed and, instead, star-shapedness has been imposed on the invariant subset. Examples are given nonexpansive but its composite fg or gf with another mapping g is nonexpansive. Then it is possible to discuss the existence of a fixed point of the composite in the framework of Chapter 2. This has been done here and, in the process, we have established the existence of a common fixed point of the composite and its constituents. The case of relative nonexpansive mappings has also been included in the discussion and an example has been provided for illustration.

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