INTRODUCTION

1. Inadequacy of classical theories: The classical linear theory of isotropic elastic solid is based on the Hooke-Cauchy law

$$t^i_j = \lambda_E e^k_k \delta^i_j + 2\mu_E e^i_j , \tag{1}$$

where t^i_j is the stress tensor, e^i_j is the infinitesimal strain tensor, λ_E and μ_E are the moduli of elasticity. Similarly, the classical theory of isotropic viscous fluid is based on the constitutive equation (Newton-Cauchy-Poisson law)

$$t^i_j = -p\delta^i_j + \lambda_V d^k_k \delta^i_j + 2\mu_V d^i_j , \tag{2}$$

p being the pressure, d^i_j the flow tensor, λ_V and μ_V are the material constants, also called the coefficients of viscosity. The theory of viscous flow based on (2) succeeded admirably in explaining the phenomena of lift, skin friction, form drag, separation, secondary flows etc. It proved to be a useful tool in the hands of hydraulic and aeronautical engineers. It failed, however, to explain the behaviour of paints, ceramics, lubricants, starch solution, rubber toluene solution, poly-iso-butylene solution in mineral oil or in tetralin, flow of colloids and suspensions, the creep of steel at high temperatures, etc.

This inadequacy of the classical theory of Newtonian fluids has led to the concept of generalized Newtonian fluids and non-Newtonian fluids. The generalization has been made in three different ways.

2. Generalization: [illegible] The first approach consists in retaining (2) and regarding λ_V and μ_V as functions of strain-rate invariants I, II, III. Non-linearity in stress-strain-rate relation is thus introduced through the coefficients of the medium. This fluid is called generalized Newtonian fluid[1].

The second approach consists in introducing in (2) more terms corresponding to the properties of the material, such as, elasticity and plasticity. This leads to the following types of non-Newtonian fluids:

i) <u>Maxwellian fluid</u>[2]: It is a viscous fluid endowed with elasticity and is characterized by the constitutive equation

$$d^i_j = \frac{\dot{p}^i_j}{2\mu_E} + \frac{p^i_j}{2\mu_V} , \tag{3}$$

where the dot denotes material derivative and p^i_j the deviatoric stress tensor. If this material is suddenly stressed and it is desired to keep the deformation constant, the stress should be gradually decreased. This is called 'relaxation of stress' and μ_V/μ_E is known as 'relaxation time'.

ii) Bingham solid: It is a material which can support a finite stress elastically without flow. It flows with constant mobility (or plastic fluidity) when the stress exceeds a certain value. In a state of flow the constitutive equation is

$$p^i_j = 2\eta d^i_j, \tag{4}$$

where

$$\eta = \eta_1 + \vartheta(2 d^i_j d^i_j)^{-\frac{1}{2}}. \tag{5}$$

The above equation has been given by Oldroyd[3]. He has shown that this material, in a state of flow, can be regarded as a viscous fluid with variable viscosity η, a scalar function of strain-rate tensor involving physical constants of the material η_1, (reciprocal mobility) and ϑ (yield stress).

The third approach consists in reformulating the stress-strain-rate relation in a more accurate manner by appealing to Stokes' principle of fluidity[4]:

'The difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned'.

Mathematically speaking, this leads to the equation

$$\bar{\bar{p}} = f(\bar{\bar{d}}), \tag{6}$$

where $\bar{\bar{p}}$ is the deviatoric stress tensor and $\bar{\bar{d}}$ is the strain-rate tensor. If $\bar{\bar{p}}$ is an isotropic function of $\bar{\bar{d}}$, the fluid is said to be isotropic. In this case the most general relation between deviatoric stress tensor and the strain-rate tensor is +

$$p^i_j = \varphi_0 \delta^i_j + \varphi_1 d^i_j + \varphi_2 d^i_\alpha d^\alpha_j. \tag{7}$$

+ For the symmetric tensors $\bar{\bar{p}}$ and $\bar{\bar{d}}$ the condition of isotropy is $\bar{\bar{p}} = k_r \bar{\bar{d}}^r$ where k_r are scalar functions of the invariants of $\bar{\bar{d}}$. With the help of Cayley-Hamilton theorem $(\bar{\bar{d}}^3 = \mathrm{III}\, 1 - \mathrm{II}\, \bar{\bar{d}} + \mathrm{I}\, \bar{\bar{d}}^2)$, $\bar{\bar{d}}^3$ and higher powers of $\bar{\bar{d}}$ can be expressed in terms of $\bar{\bar{d}}^2$, $\bar{\bar{d}}$ and 1. Hence the expression for $\bar{\bar{p}}$ for isotropic fluid reduces to $\bar{\bar{p}} = \varphi_0 1 + \varphi_1 \bar{\bar{d}} + \varphi_2 \bar{\bar{d}}^2$, which is the same as (7).

The power series for $g_r \ (r=0,1,2)$ is of the form

$$g_r = g_{rijk} \mathrm{I}^i \, \mathrm{II}^j \, \mathrm{III}^k, \tag{8}$$

where g_{rijk} are dimensional constants. A fluid obeying (7) is also called non-Newtonian fluid. Non-linearity in stress-strain-rate relation is introduced through the coefficients g_r and also through the occurrence of the term $d^i_\alpha d^\alpha_j$ in (7). This non-linearity persists even when g_r are taken as material constants. This generalization leads to two different types of fluids: (i) Stokesian fluid (ii) Reiner-Rivlin fluid.

The non-Newtonian fluid considered in different problems presented in this thesis is the Reiner-Rivlin fluid. We shall describe some characteristic features of this fluid in the following few pages.

3. Natural time: In general the coefficient of viscosity μ_v of a fluid depends on temperature and pressure. This relation can be expressed in the non-dimensional form

$$\frac{\mu_v}{\mu_{v_N}} = f\left(\frac{\theta}{\theta_0}, \frac{p}{\mu_{EN}}, \cdots\cdots\right), \tag{9}$$

where θ is the temperature, $\mu_{v_N}, \theta_0, \mu_{EN}$ are material moduli of dimensions $ML^{-1}T^{-1}, \Theta, ML^{-1}T^{-2}$ respectively and the dots stands for other possible dimensionless scalars. The ratio $\mu_{VN}/\mu_{EN} = t_n$ has dimensions of T and is called the 'natural time' of the fluid.

4. <u>Reiner-Rivlin fluid</u>[5]: This fluid possesses natural time. Its response depends on its history and it may be called a body with memory. The mathematical formulation of Stokes' principle for this fluid is

$$\bar{\bar{t}} = p f(t_n \bar{\bar{d}}, \ p t_n/\mu_{VN}, \ p/\bar{p}, \ \theta/\theta_0). \tag{10}$$

For an isotropic fluid (10) reduces to

$$p^i_j = \mu_{VN}\left(\frac{1}{t_n} g_0 \delta^i_j + g_1 d^i_j + t_n g_2 d^i_\alpha d^\alpha_j\right), \tag{11}$$

$$g_r = g_{rijk} t_n^{i+2j+3k} \, \mathrm{I}^i \, \mathrm{II}^j \, \mathrm{III}^k, \tag{12}$$

where g_{rijk} are dimensionless functions of $p/\bar{p}$, θ/θ_0 and $p t_n/\mu_{VN}$. The relation between $\mathfrak{g}_r$ and g_r can be obtained by comparing (11) with (7).

Rivlin has considered the case of incompressible fluids where $\mathfrak{g}_r$ are taken as material constants. The natural time t_n for this fluid is

$$t_\eta = \frac{g_{2000}}{g_{1000}} = \frac{2\mu_c}{\mu_v},$$

(13)

where $g_{1000} = 2\mu_v$ and $g_{2000} = 4\mu_c$. (the coefficient of cross-viscosity).

5. <u>Dissipation function</u>: The joint invariant of $\bar{\bar{p}}$ and $\bar{\bar{d}}$ which is also called dissipated power, is defined by

$$\phi(\bar{\bar{p}}, \bar{\bar{d}}) = p^i_j d^i_j = \phi(\bar{\bar{d}}, \bar{\bar{p}}).$$

(14)

Consideration of isotropy reduces it to

$$\phi(\bar{\bar{p}}, \bar{\bar{d}}) = \phi(\mathrm{I}, \mathrm{II}, \mathrm{III}) = \phi_{ijk}\, \mathrm{I}^i\, \mathrm{II}^j\, \mathrm{III}^k,$$

(15)

where

$$\phi_{ijk} = g_{0,i-1,j,k} + g_{1,i-2,j,k-1} - 2g_{1,i,j-1,k} + g_{2,i-3,j,k} + 3g_{2,i,j,k-1}.$$

(16)

Using (8) and (16), we can write (15) as

$$\phi = g_{0000} I + (g_{0100} + g_{1000}) I^2 - 2 g_{1000} II + (g_{0200} + g_{1100} + g_{2000}) I^3 + (g_{0010} - 2 g_{1100} - 3 g_{3000}) I\,II + 3 g_{2000} III + \cdots \quad (17)$$

From physical considerations, the dissipated power should always be positive for all types of flows actually taking place. Duhem and Stokes found that for Newtonian fluids ϕ reduces to a quadratic form which is positive definite, if

$$\mu_v \geqq 0, \quad 3\lambda_v + 2\mu_v \geqq 0 .$$

They proved that this condition is also sufficient. In the next approximation ϕ is given by a cubic function of strain-rate, which is not positive for all values of strain-rate tensor. Hence the inequality

$$\phi \geqq 0 \quad (18)$$

restricts the possible type of flows actually taking place in non-Newtonian fluids.

6. <u>Mean pressure:</u> For incompressible fluids $I = 0$. There is no loss of generality in assuming $g_0 = 0$, for in incompressible fluids p always occurs in the combination

$-p + \mathcal{G}_0$, which can be taken to be the unknown.

In this case the stress-strain-rate relation for isotropic fluids reduces to

$$p^i_j = \mathcal{G}_1(\mathrm{II}, \mathrm{III})\, d^i_j + \mathcal{G}_2(\mathrm{II}, \mathrm{III})\, d^i_\alpha d^\alpha_j. \tag{19}$$

From (19), we get

$$3(p - \bar{p}) = -\mathrm{II}\, \mathcal{G}_2(\mathrm{II}, \mathrm{III}), \tag{20}$$

where p is the hydrostatic pressure. For a compressible fluid p is taken to be the thermo-dynamic pressure, for an incompressible fluid p is a primitive unknown and may be taken as any convenient part of the stress. Here $\bar{p}$ denotes the mean pressure. In the case of Newtonian fluids $\mathcal{G}_2 = 0$, so $p = \bar{p}$. The statement $p = \bar{p}$ implies linear relationship between the stress tensor and the strain-rate tensor. The value of II is always negative and it has been proved by Reiner[6] that $\mathcal{G}_2 > 0$ for all fluids. Hence it can be concluded that in non-Newtonian fluids the hydrostatic pressure exceeds the mean pressure at every point where the fluid is suffering deformation.

7. <u>Kelvin and Poynting effects in Reiner-Rivlin fluid:</u> The constitutive equation (7) was proposed on purely theoretical considerations by Reiner[7] in 1945. The incompressible non-Newtonian fluid is characterized by three material constants, its density ρ , the coefficient of viscosity $\mu_v (ML^{-1}T^{-1})$ and the coefficient of cross-viscosity μ_c (ML^{-1}). Reiner[8] showed that the presence of μ_c implies that a simple shearing stress causes not only a continuous shearing of the liquid but also a 'continuous lengthening (or shortening) in the direction normal to the plane of shear'. In short the fluid exhibits Kelvin and Poynting[9] effects so well known in the theory of non-linear elasticity.

8. <u>Experimental work:</u> Experimental confirmation of Reiner's theoretical predictions about the behaviour of such fluids came from Weissenberg[10], Garner and Nissan[11], Lee and Warren[12], and Merrington[13]. Weissenberg explains these effects by attributing the property of elasticity to the fluid. Rivlin[14] has argued that the liquid in Weissenberg's experiments cannot possess elasticity and that the effect is entirely due to cross-viscosity. Truesdell[15] is of the opinion that an attempt to represent the phenomena in fluids by a theory of elasticity is inappropriate.

9. Measurement of the coefficient of cross-viscosity: The measurement of the coefficient of cross-viscosity was attempted by Greensmith and Rivlin[16] in 1952. They subjected the solution of poly-iso-butylene in tetralin to torsional motion between the parallel bases of two co-axial cylindrical cups, the inner being stationary and the outer rotating at a constant speed Ω. The surface traction normal to the base of the stationary cup was measured with the help of manometer inserted in it. They called the coefficient of cross-viscosity as the normal stress coefficient and denoted it by Ψ. They took Ψ to be function of the second invariant K_2 of the strain-rate tensor, the third invariant being zero. They found, by plotting graph of $\bar{h}_0 - \bar{h}$ against $r\Omega/l$, that the relation between them is linear for all values of $r\Omega/l$ except when $r\Omega/l$ is very small or very large. In this experiment l is the distance between the bases of the two cylinders and r is the radial distance from the centre of the inner cylinder. The mean rise of the liquid in the manometer for two directions of rotation is denoted by $\bar{h}$, while $\bar{h}_0$ is the rise of the liquid at $r = 0$. For the range of linear relationship between $\bar{h}_0 - \bar{h}$ and $r\Omega/l$ they took

$$\bar{h}_0 - \bar{h} = \alpha + \beta K_2^{\frac{1}{2}},$$

where

$$K_2 = \frac{\pi}{3600}\left(\frac{r\Omega}{l}\right)^2.$$

Putting this value of $\bar{h}_0 - \bar{h}$ in the expression for Ψ, theoretically deduced, they got

$$\Psi = \Psi' K_2^{-\frac{1}{2}},$$

where

$$\Psi' = \frac{1}{4}\rho g \beta = 15 \text{ dynes sec cm}^{-2}.$$

Plotting $(\bar{h}_0 - \bar{h})/r^2\Omega^2$ against $r^2\Omega^2$ for small values of $r\Omega/l$, they found

$$\bar{h}_0 - \bar{h} = (\alpha' - \beta' K_2) K_2,$$

where

$$\alpha' = 1{\cdot}96 \times 10^{-3} \text{ cm sec}^2,$$

$$\beta' = 7 \times 10^{-7} \text{ cm sec}^4.$$

Putting this value of $\bar{h}_0 - \bar{h}$ in the expression for Ψ, theoretically deduced, they found

$$\Psi = \Psi_0 + \Psi_1 K_2,$$

where

$$\Psi_0 = \frac{1}{3}\rho g \alpha' = {\cdot}61 \text{ dynes sec}^2 \text{ cm}^{-2},$$

and

$$\Psi_1 = -\frac{2}{5}\rho g \beta' = -3\times 10^{-4} \text{ dynes sec}^4 \text{ cm}^{-2}.$$

In 1957 the results of Greensmith and Rivlin's[16] experiment were disputed by Roberts[17]. He subjected the solution of poly-iso-butylene in tetralin to stationary torsion shearing motion in the conical cup of the type introduced by Mooney[18]. His experimental results were incompatible with the theory of Reiner[17] and Rivlin[19] and were in accordance with the theory of Weissenberg[10] and Lodge. They assumed that there are in the liquid elastic elements, which as a result of velocity gradient existing in the liquid, are subjected to finite strains. The normal stress arises as a result of these finite strains. Greensmith observed this experiment and agreed with Roberts' view. In his opinion the theory developed for his experiment on the basis of cross-viscosity did not correspond to the experiment, for the theory did not take into account the edge effects present in his experiment.

10. <u>Theoretical work:</u> Reiner[6] solved the following problems:

i) A cylindrical rod rotating in a liquid (Reiner-Rivlin).

ii) The rotating coaxial cylindrical viscometer.

iii) The tube viscometer.

iv) The parallel plate torsion viscometer.

The motivation in these problems was rheological and crucial tests were proposed to ascertain whether the observed effects were due to elasticity or cross-viscosity. He found that rubber-toluene solutions possessed cross-viscosity as well as elasticity. It appeared to him that the elastic component was more pronounced but less stable gradually disappearing in continuous shear and restored through rest, while the viscous component was apparently stable but less pronounced.

Ericksen[20] has discussed the flow of Reiner-Rivlin fluid through straight pipes of non-circular cross-section under constant pressure gradient. He finds that no rectilinear flow is, in general, possible unless body forces are applied to the fluid. For circular pipe alone rectilinear flow is possible without body forces. Green and Rivlin[21] have found that for an elliptic pipe, in addition to axial flow, secondary flows are developed in the cross-sectional planes. Similar phenomena are observed in the turbulent flow of Newtonian fluids. This led Rivlin[22] to suggest that turbulent Newtonian fluid may, for certain purpose, be regarded as non-Newtonian fluid.

11. Motivation, extent and scope of the work: The investigations reported in part I of the thesis were undertaken to extend the idea of superposability to the domain of non-Newtonian fluids. This idea was first proposed by Strang and

and developed by Ballabh[23] in the case of Newtonian fluids. This theoretical study led us to investigate the relationship between circulation preserving motions and superposable motions in non-Newtonian fluids. For Newtonian fluids, this relation was first explicitly stated by Truesdell[24]. We have proved a number of theorems on superposability and circulation preserving motions in non-Newtonian fluids. Some theorems are the extensions of their classical analogues while others are new and do not have their counterparts in the classical theory. This is due to non-linearity in the stress-strain-rate relation.

The special problems treated in part II are extensions of some classical results to non-Newtonian fluids. They are

i) Rotation of an infinite plane lamina.

ii) Rotation of two infinite plates spaced a distance d^* apart.

iii) Rotatory oscillations of an infinite plane lamina.

iv) Flow near a stagnation point (axisymmetric case).

In these problems the equations of motion of non-Newtonian fluids reduce to ordinary differential equations which can be integrated by numerical methods. It is expected that the solutions will provide useful information about the behaviour of non-Newtonian fluids, and may eventually explain some experimental results of Popper and Reiner[25], and Ward and Lord[26]. The results of the investigations are summarized in the synopsis.