1.1 Introduction :

This thesis deals with some aspects of non-convex programming and analysis. Specifically, in the first part of this thesis we have presented (i) weak, strong and converse duality results for a pair of symmetric dual nonlinear programming problems, for weaker Pseudo-convex / Pseudo-concave function with an additional feasibility condition and derived the same results for strongly Pseudo-convex/strongly Pseudo-concave function without any additional feasibility condition ; (ii) obtained similar results for a pair of mixed integer nonlinear programming problems which are non-symmetric in general but symmetric under particular conditions, from duality point of view and (iii) solved a pair of symmetric dual nonlinear programming problems for weaker strongly Pseudo-convex/strongly Pseudo-concave function, using complementarity method.

The second part of this thesis is devoted to non-convex analysis. In this part some simple yet, more general results are obtained for composite objective functions using mid-point Quasiconvexities . The familiar concepts of Logarithmic and Harmonic convexities have been generalized to semilocally Logarithmic and semilocally Harmonic convexities.

basis upon which algorithms and axioms for more general non-convex situations can be developed. Convexity thus plays much the same role as that of linearity in the study of dynamic systems.

Contribution of this thesis is in the first direction. Here, we have attempted to extend the application domain of some familiar results of convex programming and analysis to more general situations which are only close to their convex counter parts in some sense. Since this thesis deals with various types of non-convex (but close to convex in some sense) problems, it is necessary to summarize their basic definitions at least for some of the important ones. Table 1.1 and Table 1.2 presented at the end of this chapter give the summary for non-convex sets and non-convex functions respectively. They present concise definitions and a few key references for some of the important types of non-convex (but close to convex in some sense) sets/functions.

In order to put the contributions of this thesis in its correct perspective, in rest of this chapter, we present from section 1.3 to 1.7, a concise review of the literature which are only relevant to this thesis. Finally, we present a brief outline of this thesis in section 1.8.

1.3 Symmetric duality in Nonlinear Programming :

Duality principle relates to constrained minimization and maximization problems (one of which is primal and the other is dual). According to this principle the existence of a solution to one of these problems ensures a solution to the other and the extrema of the two problems are equal.

Like linear programming, duality plays a key role in nonlinear programming also. Several authors have formulated different nonlinear dual problems :

Some of the important formulations are :

(1) duality via conjugate function (Fenchel (1949), Rockafellar (1966, 1968, 1969, 1970), Whinston (1967)),

(2) duality via Lagrangian multipliers (Geoffrion (1971)),

(3) duality of the minimax type (Stoer and Witzgall (1970)) and (4) duality of symmetric type (Dantzig et al (1965), Bazaraa and Goode (1973)).

A pair of symmetric dual nonlinear programming problems has the following general formulation :

$$(P_o) \text{ (Primal) : } \operatorname{Min} \left\{ f(x,y) = K(x,y) - y^t \nabla_y K(x,y) \right\}$$

$$\text{s.t.} \quad (x,y) \in C_1 \times C_2 , \quad \nabla_y K(x,y) \in C_2^*$$

$$(D_o) \text{ (Dual) : } \operatorname{Max} \left\{ g(x,y) = K(x,y) - x^t \nabla_x K(x,y) \right\}$$

$$\text{s.t.} \quad (x,y) \in C_1 \times C_2 \;, \quad - \nabla_x K(x,y) \in C_1^*$$

where $K(x,y)$ is real-valued twice differentiable function in R^{n+m}, C_1 and C_2 are closed convex cones in R^n and R^m respectively, C_i^* $(i = 1,2)$ is the polar of C_i, $\nabla_x K(x,y)$ and $\nabla_y K(x,y)$ are gradient vectors of $K(x,y)$ with respect to x and y respectively. For convex/concave function together with some regularity condition, Bazaraa and Goode (1973) have established weak, strong and converse duality theorems for the problems (P_o) - (D_o) above. The same results have been obtained by Dantzig et al (1965) for non-negative orthant in place of closed convex cones.

In a recent paper (Mishra et al (1984)), we have studied this type of duality for more general Pseudo-convex / Pseudo-concave function. With an additional feasibility condition, we have obtained weak, strong and converse duality results. We have also derived these duality results for strongly Pseudo-convex / strongly Pseudo-concave function where the feasibility condition is not required. The details of these results are presented in Chapter II.

1.4 Minimax and Symmetric Duality in Mixed Integer Nonlinear Programming :

Let us consider the following pair of nonlinear programs which are non-symmetric in duality sense.

$$(P_o) \quad \underset{x^1}{\text{Max}} \ \underset{x^2, y}{\text{Min}} \left\{ f = K(x,y) - \lambda (y^2)^t \nabla_{y^2} K(x,y) \right\}$$

$$\text{s.t.} \quad x^1 \in U , \quad (x^2, y) \in C_1 \times T , \quad \nabla_{y^2} K(x,y) \in C_2^*$$

$$(D_o) \quad \underset{y^1}{\text{Min}} \ \underset{x, y^2}{\text{Max}} \left\{ g = K(x,y) - \mu (x^2)^t \nabla_{x^2} K(x,y) \right\}$$

$$\text{s.t.} \quad y^1 \in V , \quad (x, y^2) \in S \times C_2 , \quad - \nabla_{x^2} K(x,y) \in C_1^*$$

where λ , μ are scalars, $\lambda \geq 1$ and $\mu \geq 1$. U and V are any two arbitrary sets of integrers in R^{n_1} ($0 \leq n_1 \leq n$) and R^{m_1} ($0 \leq m_1 \leq m$) respectively . C_1 and C_2 are closed convex cones with vertices at the origin and with nonempty interiors . C_1^* and C_2^* are the polars of C_1 and C_2 respectively. $K(x,y)$ is real-valued twice differentiable function defined on an open set in R^{n+m} containing $S \times T$ where $S = U \times C_1$ and $T = V \times C_2$. $\nabla_{x^2} K(\bar{x}, \bar{y})$ and $\nabla_{y^2} K(\bar{x}, \bar{y})$ denote the gradient vectors of K with respect to x^2 and y^2 respectively at the point $(\bar{x}, \bar{y})$.

For $\lambda = \mu = 1$, the programs $(P_o) - (D_o)$ above reduce to a pair of minimax and symmetric dual mixed-integer nonlinear programs. This symmetric case is studied by Balas (1970) and Mishra and Das (1980). While Balas (1970) considered concave/convex function and non-negative orthant as the cone, Mishra and Das (1980) generalized this to any arbitrary cone.

In a recent paper (Mishra et al (1984)), with a feasibility condition, we have derived the weak, the strong and the converse duality results for Pseudo-convex/Pseudo-concave function. We have also derived the same results for strongly Pseudo-convex/strongly Pseudo-concave function where the feasibility condition is not required. These results are valid for the more general non-symmetric formulation presented above. The details of these results are given in Chapter III .

1.5 Complementarity in Symmetric Dual Non-linear Programming Over Cone :

Several problems arising in different fields(for example, mathematical programming, game theory , mechanics, plasticity and structural engineering and economic equilibrium) have the same mathematical form which may be stated as follows :

For a given map F from R^n into itself, find $x \in R^n$ satisfying

$$x \geqslant 0 , \quad F(x) \geqslant 0 , \quad x^t F(x) = 0$$

If $F(x) = Mx + b$ where M is an $n \times n$ matrix and b is an n-vector, the above problem is referred as linear complementarity problem (LCP) , otherwise it is called nonlinear complementarity problem (NCP) .

The above problem can also be restated in the following manner :

Find $x \in R^n$ such that

$$x \in R^n_+ , \quad F(x) \in R^n_+ , \quad x^t F(x) = 0$$

Since R^n_+ is a cone in R^n (indeed a polyhedral cone) the above formulation motivated several authors (see, for example, Habetler and Price (1971)) to define general complementarity problem over a closed convex cone and its polar. This formulation is as follows :

Find $x \in R^n$ such that

$$x \in S , \quad F(x) \in S^* , \quad x^t F(x) = 0$$

where S is a closed convex cone and S^* is its polar.

Dantzing and Cottle (1967) have solved symmetric dual programs by complementarity method over non-negative orthant. Craven and Mond (1977) have generalized these results to arbitrary cone. In both the cases, they have considered convex/concave function . But we have solved symmetric dual program with weaker strongly Pseudo-convex / strongly Pseudo-concave function. Also we have solved the same problem for Pseudo-convex/Pseudo-concave function with an additional feasibility condition. These results are presented in Chapter IV.

1.6 Nonconvex and Composite Objective Functions :

A composite function $F = fou$ where f is a real function in m variables and u is a vector - valued function in n variables is of particular interest to economists and operations researchers. In some areas of economics (e.g. in social welfare functions) and operations research (e.g. in multiple objective programming) , the problem of particular interest is to choose f for any feasible u such that the function $F = fou$ has the same nature as that of f .

Bereanu (1964) has considered this problem and proved that $F = fou$ is convex for every vector function u with the range in the domain of f and having its components, separately convex or concave iff f is convex and partially monotone, increasing in the convex components and decreasing in the concave components. Mangasarian (1970) has generalized this to Pseudo-convex and Quasi - convex functions. Bereanu (1972) in addition to considering strictly Quasi-convex functions has also given some important applications. In this thesis we have obtained some simple, yet more general results on composite objective functions under m-Quasiconvexity , strictly m-Quasiconvexity and r- Quasi convexity assumptions. These results are presented in Chapter V .

1.7 Semilocally Logarithmic and Semilocally Harmonic Convexities :

The concept of convexity for functions is generalized to semilocally convexity by Ewing (1977) . Recently Kaul and Kaur (1982) have introduced the concepts of semilocally Pseudo-convex and semilocally Quasiconvex functions. We have generalized Logarithmic and Harmonic convexities to semilocally Logarithmic and semi-locally Harmonic convexities respectively. This is presented in Chapter VI .

1.8 A Brief Outline of the Thesis :

For the purpose of presentation, we have divided the rest of this thesis into six chapters. A chapterwise summary is given below.

In Chapter II, we have solved a pair of symmetric nonlinear programming problems. In this chapter we have presented two different types of results. Firstly for more general Pseudo-convex / Pseudo-concave function, we have obtained weak duality and related results under an additional feasibility condition. Secondly, for strongly Pseudo-convex / strongly Pseudo-concave function, we have established the same results without any feasibility assumption.

In Chapter III we have formulated a pair of maximini and minimax nonlinear mixed integer programming problems which are nonsymmetric in duality sense, but in particular reduce to the symmetric case. Firstly, we have solved this problem for Pseudo-convex / Pseudo-concave function with an additional feasibility condition. Secondly, we have solved this problem for strongly Pseudo-convex / strongly Pseudo-concave function without any feasibility condition.

In Chapter IV we have solved symmetric dual programs by using complementarity method over arbitrary cone. Our main results are motivated by the works of Dantzig and Cottle (1967), Craven and Mond (1977) who proved our results for stronger (convex/concave) function. Our results are valid under weaker (strongly Pseudo-convex / strongly Pseudo-concave) assumptions. These results are

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also valid for Pseudo-convex/Pseudo-concave function with an additional feasibility condition.

We have devoted Chapter V to the study of composite objective functions. In this chapter, some results on composite objective functions have been obtained for midpoint Quasiconvex, strictly midpoint Quasiconvex and r-Quasiconvex functions.

In Chapter VI , we have generalized Harmonic convexities and Logarithmic convexities to semilocally Harmonic convexities and semilocally Logarithmic convexities respectively and have studied some of its basic properties.

Finally, in Chapter VII , we have summarized the contributions of this thesis and indicated some areas of future research relevant to the topics covered in this thesis.