Chapter 1

Introduction

Here we present some basic concepts, definitions and governing equations that are standard in several text books on Fluid Mechanics and Flow through Porous Media. Some of these are based on the following literature (Please see Batchelor, 2000; Plawsky, 2001; Basmadjian, 2004; Kohr and Pop, 2004; Nield and Bejan, 2006).

1.1 Porous Media

Porous media may be defined as a matrix with pores either connected or non-connected dispersed in the medium in a regular or random manner. The porous media consist of two phases, one the solid phase which can be termed as *solid matrix* and the other which is not the part of solid matrix is termed as *void space (pore space)*. Some of the pores in the pore space may be interconnected and this pore space is generally referred as the *effective pore space* and the non–connected pore space may be considered as part of the solid matrix. In addition, porous material is classified as *ordered porous material* and *random porous material* depending on whether the pore spaces inside the porous material are ordered or disordered, respectively. Examples of porous materials are numerous. Soils, ceramics, fibrous aggregates, filter papers, sand filters and a loaf of bread are just a few examples.

Porosity of a porous material is defined as the fraction of void space to the total volume with respect to a given material. This gives the total porosity of the medium. The fluid may not be able to move if the voids are not interconnected, as the structure may appear like closed, isolated pores. The degree to which pores within the material are interconnected is known as *effective porosity*. This is while excluding the non–connected pores. Thus, effective porosity is typically less than the total porosity. Porosity represents the storage capacity of the material. The permeability k is the most important physical property of a porous medium like the porosity is its most important geometrical property. Permeability is a measure of the ease with which fluid flows through a porous material. Permeability is often directional in nature. The characteristics of the interstices of certain materials may cause the permeability to be significantly greater in a particular direction.

1.2 Transport Phenomena

The exchange of momentum, mass or energy between various systems of engineering interest or the exchange of mechanical and thermal properties of material are the key issues of classical transport phenomena. This describe the processes that take a system of particles from a non–equilibrium state to an equilibrium state or from an equilibrium state to a non–equilibrium state or from one non–equilibrium state to another. The exchange between two elements of matter with different properties makes the amount of some quantity satisfying a conservation law associated with one element to decrease while the amount associated with the other to increase. Transport phenomena is a fundamental component of disciplines related with fluid motion, heat transfer and mass transfer. Here we discuss some of the transport processes.

1.2.1 Momentum Transfer

While dealing with mass transfer, the fluid is considered to be a continuous distribution of matter. Random diffusion of molecules cause transfer of momentum in various directions. When a fluid is flowing along a particular direction (say x-axis) parallel to a solid surface element, due to molecular motions and interactions the momentum directed along the flow direction is transferred in other direction (say z-axis) from the faster to the slower moving layer. This is a simple shearing motion in which planes of fluid parallel to the surface element slide rigidly over one another. The equation of the momentum transport with respect to the axes referred above can be written as

$$\tau_{zx} = -\nu \frac{\partial(\rho v_x)}{\partial z}, \tag{1.2.1}$$

where τ_{zx} is the flux of x-directed momentum in the z direction, $\nu = \mu/\rho$ is the momentum diffusivity, ρ the density. The transport of momentum constitutes internal friction and a fluid exhibiting internal friction is said to be viscous. The velocity gradient appearing in the above momentum transport equation can be thought of as a driving force.

1.2.2 Mass Transfer

There are two mechanisms in general, by which mass transfer takes place, one convection and the other diffusion. Material transportation due to the mean motion of the fluid in which it is carried is called convection and that due to the random motion of the molecules within the fluid, even in the absence of any mean flow is called diffusion. Mass transport occurs from a high concentration $\mathbf{c}$, to a low concentration minimizing any concentration difference within a system. The greater the difference in concentration per unit distance, x, the larger the transport of mass. This can be expressed in terms of the molar concentration gradient $d\mathbf{c}/dx$ with the proportionality constant D is known as the (mass) diffusivity of the species and the corresponding mathematical representation is nothing but the Fick's law of diffusion given by

$$\begin{aligned} \text{Molar flow } N\ (mol/s) &= -DA_r\frac{d\mathbf{c}}{dx} \\ \text{Molar flux } N/A_r\ (mol/sm^2) &= -D\frac{d\mathbf{c}}{dx}, \end{aligned} \tag{1.2.2}$$

where A_r is the cross sectional area.

1.2.3 Concentration Equation

For a multi component mixture, the species balance equation is governed by Stefan-Maxwell equation given by (Bird et al., 2002)

$$\nabla x_A = \sum_{\substack{B=1 \\ B\neq A}}^{B=N} \frac{x_A(\mathbf{c}_B v_B) - x_B(\mathbf{c}_A v_A)}{D_{AB}}, \qquad A = 1, 2, ...N-1 \tag{1.2.3}$$

where $x_A,\ x_B$ are mole fractions of species, $\mathbf{c}_A,\ \mathbf{c}_B$ are concentration of species, $v_A,\ v_B$ are velocities of species and D_{AB} is the binary diffusivity of the corresponding system. It may be noted that under constant mass density assumption, the species balance equation (1.2.3) reduces to

$$\frac{\partial \mathbf{c}_A}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{c}_A = D\nabla^2\mathbf{c}_A \pm k_r\mathbf{c}_A^n, \tag{1.2.4}$$

where D is diffusivity of the medium, k_r is the rate of reaction and n is the order of reaction $(n = 0,\ 1, ...)$. The given convection–diffusion–reaction equation (1.2.4) is usually used for diffusion in dilute liquid solution at constant temperature and pressure.

It is seen in literature that the reaction rate term, i.e., the second term in the right hand side of (1.2.4), can be modeled as

$$k_r \mathbf{c}_A^n = \frac{k_1 \mathbf{c}_A}{k_2 + \mathbf{c}_A}. \tag{1.2.5}$$

Further, one can obtain the following limiting models, namely

Zero Order:

$$\frac{k_1 \mathbf{c}_A}{k_2 + \mathbf{c}_A} = k_1, \quad \mathbf{c}_A >> k_2 \tag{1.2.6}$$

First Order:

$$\frac{k_1 \mathbf{c}_A}{k_2 + \mathbf{c}_A} = \frac{k_1}{k_2} \mathbf{c}_A, \quad \mathbf{c}_A << k_2. \tag{1.2.7}$$

The above zero and first order reactions correspond to the case of $n = 0$ and $n = 1$, respectively in equation (1.2.4).

1.3 Equation of Motion

1.3.1 Conservation of Mass (Equation of Continuity)

The flow equations for a physical system are based on conservation principles that are fundamental. These principles can be expressed in terms of mathematical terms which are usually partial differential equations. It is well known that when a region of a fluid contains neither sources nor sinks, the principle of conservation of mass tells that the amount of fluid within the region is conserved. The corresponding mathematical representation is called equation of continuity, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.3.1}$$

where ρ is the density of the fluid and $\mathbf{v}$ is the area averaged velocity vector. For an incompressible fluid, the density of any particle is invariable with time so that $\frac{\partial \rho}{\partial t} = 0$. For a homogeneous and incompressible fluid ρ is constant throughout the entire fluid and hence the equation of continuity reduces to

$$\nabla \cdot \mathbf{v} = 0. \tag{1.3.2}$$

A vector $\mathbf{v}$ having zero divergence is said to be *solenoidal or divergence–free*.

1.3.2 Viscous Flow

The equations describing the motion of incompressible Newtonian flow are the equation of continuity

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.3}$$

together with the Navier–Stokes equations, which are nonlinear partial differential equations given by

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}, \tag{1.3.4}$$

where $\mathbf{v}$ and p are the velocity and pressure fields of the flow, ρ and μ are the density and dynamic viscosity of the fluid, and $\mathbf{g}$ is the body force per unit mass of the fluid. In terms of the stress tensor $\mathbb{T}$, whose components are given by

$$T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \tag{1.3.5}$$

the equation (1.3.4) can be written in the form

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \mathbb{T} + \rho \mathbf{g}. \tag{1.3.6}$$

In the above stress tensor, δ_{ij} is the Kronecker delta function. One can define the modified pressure P and $\mathbb{T}^{mod}$, the modified stress tensor given by

$$P = p - \rho(\mathbf{g} \cdot \mathbf{x}) + a_0, \tag{1.3.7}$$

$$T_{ij}^{mod} = -P\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \tag{1.3.8}$$

where

$$E_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

are the components of the rate of strain tensor and a_0 is an arbitrary constant that is real. Considering this modification, the Navier–Stokes equations may be written as

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \mathbb{T}^{mod}. \tag{1.3.9}$$

Hence, without loss of generality, the equation of continuity and Navier–Stokes equations may be written as follows

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.10}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \mu \nabla^2 \mathbf{v} = \nabla \cdot \mathbb{T}. \tag{1.3.11}$$

The following non–dimensional variables are introduced : $\mathbf{v}^{'} = \dfrac{\mathbf{v}}{U_\infty}$, $\mathbf{x}^{'} = \dfrac{\mathbf{x}}{L}$, $P^{'} = \dfrac{P}{\mu \frac{U_\infty}{L}}$, $t^{'} = \dfrac{t}{t_c}$ and the above equations on dropping the prime reduce to

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.12}$$

$$\frac{\rho \mu L^2}{t_c} \frac{\partial \mathbf{v}}{\partial t} + \frac{\rho \mu U_\infty}{L} \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \mu \nabla^2 \mathbf{v}. \tag{1.3.13}$$

The dimensionless factors multiplying the left hand side terms can be defined as $T_u = \dfrac{\rho \mu L^2}{t_c}$ is the unsteadiness parameter and $Re = \dfrac{\rho \mu U_\infty}{L}$ is the Reynolds number.

When both the parameters T_u and Re are small (i.e., $T_u << 1$, $Re << 1$), this is the case of viscous terms dominating the inertial terms and one can neglect the left hand side terms and in terms of dimensional variables, we have

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.14}$$

$$\mu \nabla^2 \mathbf{v} = \nabla P, \tag{1.3.15}$$

which are called Stokes equations. If $Re << 1$ but $T_u \sim 1$, then the nonlinear inertial term on the left hand side can be neglected and correspondingly in terms of dimensional variables, we have

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.16}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \mu \nabla^2 \mathbf{v}, \tag{1.3.17}$$

which are called the unsteady Stokes equations or the linearized Navier–Stokes equations.

If the flow is oscillatory in nature with frequency ω, the velocity and pressure fields $\mathbf{v}$ and P may be set as $\mathbf{v}=\mathbf{V}e^{-i\omega t}$ and $P = pe^{-i\omega t}$. Thus, the governing equations given in (1.3.16)–(1.3.17) transform to

$$\nabla \cdot \mathbf{V} = 0, \tag{1.3.18}$$

$$-i\rho\omega \mathbf{V} = -\nabla p + \mu \nabla^2 \mathbf{V}. \tag{1.3.19}$$

These are the governing equations for oscillatory Stokes flow. We realize that these equations are mathematically similar to Brinkman equations used in porous media and we postpone this discussion until we introduce the governing equations corresponding to flow through porous media.

The governing equation of laminar flow through homogeneous porous media is based on a classical experiment performed by Darcy (1856). A common application of this is groundwater flow through an aquifer. The outcome of this experiment is the popularly known Darcy's law which is an empirical relationship among the flow rate (Q), the cross–sectional area (A_r) of the aquifer perpendicular to the flow, the hydraulic gradient ($\triangle h$/L), and the hydraulic conductivity of the aquifer (k^*), where L is the length of the aquifer. Darcy's law can be expressed mathematically as

$$Q = \frac{k^* A_r \triangle h}{L}. \tag{1.3.20}$$

The pressure head h is equal to $(z + \dfrac{p}{\rho g})$, where z is the elevation, p is the pressure and ρ is the density of the fluid. Experiments show that the constant of proportionality k^* is the hydraulic conductivity which is proportional to the density and inversely to the viscosity of the fluid. Expressing the above equation in terms of the space averaged velocity (or Darcian velocity), we have

$$v = -\frac{k}{\mu}\frac{d}{ds}(p + \rho g z), \tag{1.3.21}$$

where k is the permeability of the medium and is given by $k = \dfrac{\mu k^*}{\rho g}$. The negative sign in the above equation indicates that the fluid velocity is in the opposite direction of increasing pressure gradient. In the case of beds of particles or fibres, the relation between the permeability k and the porosity Φ of the medium is given by $k = \dfrac{d^2\Phi^3}{180(1-\Phi)^2}$, where d is the effective average particle or fibre diameter. This relation is known as Carman–Kozeny relation (Nield and Bejan, 2006). The constant 180 in the mentioned relation was obtained by seeking a best fit with experimental results. The macroscopic flow equation given in (1.3.21) is one-dimensional, and hence it is essential to generalize this result to a vector form. Thus in a three dimensional flow system, the resultant velocity at any point is directly proportional to the resultant pressure gradient. Therefore, the general form of the Darcy's law is

$$\mathbf{v} = -\frac{k}{\mu}(\nabla p - \mathbf{f}), \tag{1.3.22}$$

where, $\mathbf{f}$ is the body force which can be replaced by the gravitational force $\mathbf{g} = (0, 0, -g)$. The permeability k is constant for an isotropic medium. This model does not take inertial effects into consideration, hence is valid for seepage velocity only, i.e., for flows with $O(Re) < 1$, where Re is the Reynolds number.

It may be noted that the Darcy's law is a first order equation where the viscous shearing stresses acting on a volume element of fluid have been neglected. The damping force, $\left(\frac{\mu v}{k}\right)$, due to the porous media has been retained is a reasonable approximation for only small permeability. On the other hand, in the context of mixture theory Rajagopal (2007) has listed the approximations under which Darcy's law is derived. Some of the major assumptions that lead to Darcy's equation are that the balance of linear momentum of the solid can be ignored, the frictional effects within the fluid due to its viscosity are neglected and the only forces are due to the interaction of the fluid with the boundaries of the pores. For more details please refer (Rajagopal and Tao, 1995; Atkin and Craine, 1976b,a). Moreover, the Stokes–Darcy coupling at a porous–liquid interface fails to handle stress boundary conditions. In order to address this issue, Brinkman (1949) suggested an equation balancing the forces acting on the volume element, i.e, pressure gradient, the divergence of the viscous stress tensor and the damping force caused by the porous mass. Brinkman equation for the fluid flow in porous media valid for large permeability is

$$\nabla p = -\frac{\mu}{k}\mathbf{v} + \mu_{eff}\nabla^2\mathbf{v}, \tag{1.3.23}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3.24}$$

where $\mathbf{v}$ is the rate of flow, μ is the fluid viscosity and μ_{eff} is the effective viscosity. The so–called effective viscosity is different from the viscosity of the fluid and is a parameter, that allows for matching the shear stress boundary condition across the free fluid–porous medium interface. But, in most of the cases it is assumed that $\mu_{eff} = \mu$. This equation has the advantage of approximating Darcy's law for small values of k and Stokes equation for large values of k. One can observe that the Brinkman equation given in (1.3.23) has a similar form as the governing equations for oscillatory Stokes flow shown in (1.3.19) except that the meaning of the physical parameters involved are different. Hence, the same mathematical theory can be adopted to both the equations, however, depending on the context.

The linearity of Stokes equations allows one to develop analytical solutions for circular / spherical geometries either in primitive variable formulation or using stream function approach. One may refer (Happel and Brenner, 1983; Padmavathi et al., 1998) for a brief idea on the existing solutions of Stokes equations. However, in order to deal with arbitrary Stokes flow past spherical geometries, the following representation

$$\mathbf{v} = \nabla \times \nabla \times (\mathbf{x}A) + \nabla \times (\mathbf{x}B), \tag{1.3.25}$$

$$p = p_0 + \mu \frac{\partial}{\partial r}(r\nabla^2 A), \tag{1.3.26}$$

of the velocity and pressure corresponding to a Stokes flow appears to be more useful (Palaniappan, 1993; Padmavathi et al., 1998; Raja Sekhar and Amaranath, 1996). In the above representation, p_0 is a constant, $\mathbf{x}$ is the position vector, A and B are scalars satisfying the equations

$$\nabla^4 A = 0, \quad \nabla^2 B = 0. \tag{1.3.27}$$

The above representation is proposed by Palaniappan et al. (1992) and later is shown to be a complete general solution of Stokes equations by Padmavathi et al. (1998).

While discussing the problem of Stokes flow past porous regions using Brinkman equation, Higdon and Kojima (1981) constructed the solutions for exterior and interior regions of porous particles. A cartesian tensor solution was also given by Yu and Kaloni (1988). Padmavathi et al. (1993) gave a solution for the velocity and pressure of Brinkman equation, considering $\mu_{eff} = \mu$, in terms of two scalar functions A and B, where

$$\mathbf{v} = \nabla \times \nabla \times (\mathbf{x}A) + \nabla \times (\mathbf{x}B), \tag{1.3.28}$$

$$p = p_0 + \mu \frac{\partial}{\partial r}[r(\nabla^2 - \lambda^2)A], \tag{1.3.29}$$

where $\lambda^2 = \frac{1}{k}$, p_0 is a constant, A and B satisfy the equations

$$\nabla^2(\nabla^2 - \lambda^2)A = 0, \quad (\nabla^2 - \lambda^2)B = 0. \tag{1.3.30}$$

Raja Sekhar et al. (1997) proved that this is a complete general solution of Brinkman equation.

As mentioned earlier, since Stokes equations in case of oscillatory flow are mathematically similar to the Brinkman equations except for the meaning of parameter λ, the representation given in (1.3.28)–(1.3.30) can also be adopted while solving problems involving oscillatory Stokes flow. Also, similar representation in case of unsteady Stokes flow can be seen in (Venkatalaxmi et al., 2004).

1.4 Boundary Conditions

Many of the physical problems are modeled in terms of partial differential equations (PDE's) as governing system of equations. Once the governing equations for a particular physical problem are known, in order to get a unique solution of the physical problem one has to specify suitable boundary conditions. Hence, the boundary value problems involving PDE's demand the unknown function or its partial derivatives to be prescribed on the boundary. Thus, it is really important in posing appropriate boundary conditions for the system under consideration. In fluid mechanical problems, the boundaries need not be impermeable walls confining the fluid to a given region in space, they may be variety of geometrical surfaces, permeable bodies. On such boundaries, one has to prescribe the unknown or gradient of the unknown or a linear combination of both. We now review such a scenario in case of transport phenomena at a porous–liquid interface.

Transport phenomena at a porous–liquid interface is seen in a variety of applications such as flow in rivers, packed bed heat exchangers, catalyst bio–reactors, nuclear waste repositories, drying processes, ground water pollution and many more. Prescribing appropriate boundary conditions at a porous liquid interface is still a challenging task for the researchers. There are three levels of descriptions namely, microscopic, mesoscopic and macroscopic in order to understand the interface region. While dealing with micro and mesoscopic levels, scalings need special attention, macroscopic scale is the most commonly used in order to study problems of practical applications. Because in this description one can adopt two homogeneous regions with a discontinuity at the interface where appropriate boundary conditions are to be specified. A detailed discussion on the suitable boundary conditions at a porous–liquid interface is given by Chandesris and Jamet (2006).

It is well known that when the momentum transport in porous medium is described by Darcy's law and the clear fluid is described by Stokes equation, the obvious boundary condition at the permeable interface is the continuity of the normal velocity, which is a consequence of incompressibility. The vanishing of the tangential velocity of the free fluid is true for impermeable surface but is not satisfactory for a permeable surface. Beavers and Joseph (1967) have shown that the wall permeability implies a non–zero velocity at the interface, i.e., an apparent slip velocity. This empirical slip flow condition is given for a rectilinear flow of a viscous fluid through a two dimensional parallel channel formed by an impermeable upper wall and a permeable lower wall as

$$\frac{\partial u}{\partial y}\mid_{y=O_+} = \gamma^*(u_B - u_D), \tag{1.4.1}$$

where O_+ is a boundary limit point from the exterior fluid, u_D is the velocity in the porous region and u_B is the velocity in the interface, γ^* is the adjustable coefficient which can depend on material properties like viscosity, μ of the fluid, permeability, k of the material. The above condition informs that the difference between the slip velocity of the free fluid and the tangential component of the seepage velocity is proportional to the shear rate of the free fluid. It can be seen that γ^* has dimensions of $(length)^{-1}$ and it may be noted that μ is the only independent quantity containing dimensions of mass and time. This suggests that γ^* is independent of viscosity, μ. Since permeability of the material characterizes a length scale $\sqrt{k}$, one can express γ^* as $\alpha/\sqrt{k}$, where α is a dimensionless quantity depending on the material parameters which characterizes the structure of permeable material within the boundary region.

Saffman gave a theoretical justification for the slip condition of Beavers and Joseph and proposed that

$$u_B = c\sqrt{k}\frac{\partial u}{\partial y}\mid_{y=O_+}. \tag{1.4.2}$$

The exact value of the slip velocity depends on the location of the interface, if the interface is shifted through a distance $s\sqrt{k}$, where s is an increment in c.

Neale and Nader (1974), as well as Vafai and Thiyagaraja (1987), and Vafai and Kim (1990), have questioned the validity of 'slip velocity' and proposed that both the velocity and shear stress must be continuous at the interface, i.e.,

$$u^l = u^p, \tag{1.4.3}$$

$$\mu\frac{du^l}{dy} = \mu_{eff}\frac{du^p}{dy}, \tag{1.4.4}$$

where u^l and u^p denote the x–component of velocity in free fluid and porous regions respectively. To incorporate this extra boundary condition, these authors employed the Brinkman's extended Darcy model to describe the flow through the porous media. The spatial changes of the local porous structure characterize the interface region. Ochoa-Tapia and Whitaker (1995a,b) reasoned that in this boundary region, the length–scale constraints, used for the derivation of the macroscopic conservation equations in both homogeneous fluid and porous regions may not be satisfied. In order to handle this difficulty in the context of volume averaging, they have developed the momentum transfer condition at the boundary between a porous medium and a homogeneous fluid as a jump condition. This is known as *stress jump boundary condition*. They used a sophisticated technique based on the non–local form of the volume averaged momentum equation in the whole domain including the heterogeneous transition region. Based on the methodology proposed by Ochoa-Tapia and Whitaker (1995a,b), the simplified form of the stress–jump boundary condition at the interface between the two homogeneous regions is given by

$$\frac{1}{\Phi}\frac{\partial u^p}{\partial y} - \frac{\partial u^l}{\partial y} = \frac{\beta}{\sqrt{k}}u^p, \tag{1.4.5}$$

where β is called stress–jump coefficient that is to be determined experimentally. In their studies, Ochoa-Tapia and Whitaker (1995a,b) have reported that β can be either positive or negative, but must be of O(1), however, the predictions have been refined in Valdés-Parada et al. (2007). In the above condition, Ochoa-Tapia and Whitaker (1995a,b) have used the relation $\mu_{eff} = \mu/\Phi$, where μ_{eff} is the effective viscosity, μ is the viscosity of the fluid, Φ is the porosity and k is the permeability of the homogeneous portion of the porous region. Kuznetsov (1996, 1998) used this stress–jump boundary condition at the interface between a porous medium and a clear fluid, to discuss flow in channels partially filled with porous medium and showed that the velocity at the interface, which is called the interfacial velocity, has wide variations with respect to changes in β in the range of -0.8 to 0.8. Also Bhattacharyya and Raja Sekhar (2004, 2005); Partha et al. (2005, 2006) have employed this stress jump condition while dealing with viscous flows through porous media. Recently, Valdés-Parada et al. (2009) have computed the stress jump coefficient where it is shown that the jump coefficient explicitly depends on porosity and Darcy number. It is also shown that for small enough Darcy number (i.e., $Da \sim 10^{-7} - 10^{-10}$), the stress jump coefficient is given by (jump coefficient=$\frac{1}{\sqrt{\text{porosity}}}$) and it takes positive values. However in the present thesis, we have considered both positive and negative values. The studies taking jump into account play a significant role and influence the physical interpretation of the solution of the problem.

The determination of jump coefficient (β) is a difficult task. However, Goyeau et al. (2003) have computed the jump coefficient by introducing the one domain approach, where the porous medium is regarded as a pseudo–fluid, and the inter–region is treated as a continuum where average coefficients are position dependent. They concluded that β is a function of velocity profile, as well as the spatial variations of the permeability and porosity. Recently, Chandesris and Jamet (2006) used the method of matched asymptotic expansions to solve the one domain approach. They proposed that the stress jump condition is related not only to the slip velocity but also to the pressure gradient through a second jump coefficient. The variations of the porosity and permeability in the transition region were expressed through two excess quantities. The jump coefficient, β proposed by Ochoa-Tapia and Whitaker (1995a,b) is a combination of the bulk and Brinkman stress contributions. In addition, a surface jump coefficient is introduced to account for the excess of surface stress at the dividing surface, which is negligible with respect to β depending on the physical configuration of the problem. While the exact set of boundary conditions that are to be used at a porous–liquid interface is still a potential topic for researchers, a review on this may be found in (Nield, 2009). A single-domain Darcy–Brinkman model is used by Le Bars and Worster (2006), to particular configurations like Poiseuille flow in a fluid overlying a porous layer, corner flow in a fluid overlying a porous layer, solidification in a corner flow. They compared this approach with the multiple domain formulation using previously available boundary conditions. They observed that when the existing interfacial conditions are used, the agreement between these two approaches is not satisfactory. A viscous transition zone is defined inside the porous domain, where the Stokes equation is still valid. Across this transition zone, they imposed continuity of pressure and velocity. They observed that this new boundary condition that involve the transition zone length as a parameter, gives better agreement between the two approaches. One has to consider various flow configurations admitting analytical solution and test this new approach proposed by Le Bars and Worster (2006), in order to better understand the transport phenomena at a porous–liquid interface.

1.5 Faxén's Laws

The expressions for the drag and torque acting on a rigid sphere of radius $'a'$ in an unbounded arbitrary Stokes flow are provided by Faxén's laws (Faxén, 1924). These expressions, on $r = a$, are as follows

$$\mathbf{F} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[T_{rr}^{en}\hat{\mathbf{e}}_r + T_{r\theta}^{en}\hat{\mathbf{e}}_\theta + T_{r\varphi}^{en}\hat{\mathbf{e}}_\varphi\right] r^2 \sin\theta \, d\theta \, d\varphi, \tag{1.5.1}$$

$$\mathbf{T} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[rT_{r\theta}^{en}\hat{\mathbf{e}}_\varphi - rT_{r\varphi}^{en}\hat{\mathbf{e}}_\theta\right] r^2 \sin\theta \, d\theta \, d\varphi, \tag{1.5.2}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\varphi$ are the unit vectors corresponding to the spherical coordinates (r, θ, φ), and T^e_{rr}, $T^e_{r\theta}$ and $T^e_{r\varphi}$ are the normal, tangential and azimuthal stress components acting on the surface of the sphere $r = a$ respectively and are given as

$$T_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}, \tag{1.5.3}$$

$$T_{r\theta} = \mu \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right], \tag{1.5.4}$$

$$T_{r\varphi} = \mu \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} + \frac{\partial v_\varphi}{\partial r} \right]. \tag{1.5.5}$$

In the next two chapters we have discussed the problem of an arbitrary Stokes flow past a porous sphere in an oscillatory flow where the flow inside the porous sphere is governed by Darcy or Brinkman equation. In each case, we derived the corresponding expressions of Faxén's laws. These expressions enable us to calculate the drag force **F** and torque **T** directly from the basic velocity corresponding to a given flow .

1.6 Fundamental Solutions

1.6.1 Steady Stokes Equation

Among few available methods to solve Stokes flow related problems, one is boundary value problem method and the other is method of fundamental solution also known as singularity method. For applications of fundamental solutions of Stokes equations in singularity and boundary integral methods, one may refer (Pozrikidis, 1992; Kohr and Pop, 2004). Fundamental solutions are computed in terms of Green's functions. The fundamental solution of Stokes flow due to a point force is obtained in terms of Green's function. The singularly forced Stokes equation together with equation of continuity is written as

$$\nabla \cdot \mathbf{v} = 0, \tag{1.6.1}$$

$$-\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{m}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0}, \tag{1.6.2}$$

where $\mathbf{m}$ is a constant vector representing the strength of the point force, and δ is the three– dimensional Dirac delta–function and $\mathbf{y}$ is the location of the singularity. Stokeslet is a singularity, which represents the Stokes flow due to a point force in free space.

We now introduce the Green's function. The velocity and pressure of Stokeslet in terms of the corresponding tensor notation in $\mathbb{R}^3$, are given by

$$\begin{aligned} v_j(\mathbf{x}) &= \frac{1}{8\pi\mu}\mathcal{G}_{jk}(\mathbf{x}-\mathbf{y})m_k, \\ p(\mathbf{x}) &= \frac{1}{8\pi}\mathbf{\Pi}_k(\mathbf{x}-\mathbf{y})m_k, \quad j,k=1,2,3. \end{aligned} \tag{1.6.3}$$

The components of the fundamental Stokes tensor $\mathcal{G}$ and those of its associated pressure vector $\mathbf{\Pi}$ determining the fundamental solution $(\mathcal{G},\mathbf{\Pi})$ of the Stokes system in $\mathbb{R}^3$, are given by (Kohr and Pop, 2004)

$$\begin{aligned} \mathcal{G}_{jk}(\mathbf{x}-\mathbf{y}) &= \frac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^3}, \\ \Pi_j(\mathbf{x}-\mathbf{y}) &= 2\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^3}. \end{aligned} \tag{1.6.4}$$

It may be noted that *Curl* of the above Stokeslet is called Rotlet, which is also a singular solution of Stokes equations. One can also see Stokeslet in two–dimensional case (Pozrikidis, 1997), but we do not review here as the present thesis deals with Stokeslet in case of three dimensions only.

1.6.2 Oscillatory Stokes Equation

The free space fundamental solution of oscillatory Stokes equations are called oscillatory Stokeslet. The singularly forced oscillatory Stokes equation together with equation of continuity in dimensionless form is written as

$$\nabla\cdot\mathbf{v} = 0, \tag{1.6.5}$$

$$-\nabla p + (\nabla^2-\lambda^2)\mathbf{v} + \mathbf{m}\delta(\mathbf{x}-\mathbf{y}) = \mathbf{0}, \tag{1.6.6}$$

where $\lambda^2 = -\frac{i\omega a^2}{\nu}$. The flow due to an oscillatory point force located at the point $\mathbf{y}$ in free space, whose strength is given by the real or imaginary part of $\mathbf{m}\exp^{-i\omega t}$, where $\mathbf{m}$ is a constant vector, is called oscillating Stokeslet. The velocity and pressure of such an oscillatory Stokeslet in terms of the corresponding tensor notation in $\mathbb{R}^3$, is given by

$$\begin{aligned} v_j(\mathbf{x}) &= \frac{1}{8\pi\mu}\mathcal{G}^{\lambda^2}_{jk}(\mathbf{x}-\mathbf{y})m_k, \\ p(\mathbf{x}) &= \frac{1}{8\pi}\mathbf{\Pi}^{\lambda^2}_k(\mathbf{x}-\mathbf{y})m_k, \quad j,k=1,2,3. \end{aligned} \tag{1.6.7}$$

The components of the fundamental oscillatory Stokes tensor $\mathcal{G}^{\lambda^2}$ and those of its associated pressure vector $\mathbf{\Pi}^{\lambda^2}$, which determine the fundamental solution $(\mathcal{G}^{\lambda^2}, \mathbf{\Pi}^{\lambda^2})$ of the oscillatory Stokes system in $\mathbb{R}^3$, are given by (Kohr and Pop, 2004)

$$\begin{aligned} \mathcal{G}^{\lambda^2}_{jk}(\mathbf{x}-\mathbf{y}) &= \tfrac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|} A_1(\lambda|\mathbf{x}-\mathbf{y}|) + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^3} A_2(\lambda|\mathbf{x}-\mathbf{y}|), \\ \Pi^{\lambda^2}_j(\mathbf{x}-\mathbf{y}) &= 2\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^3}, \end{aligned} \tag{1.6.8}$$

where

$$\begin{aligned} A_1(\mathcal{R}) &= 2e^{-\mathcal{R}}(1+\mathcal{R}^{-1}+\mathcal{R}^{-2}) - 2\mathcal{R}^{-2}, \\ A_2(\mathcal{R}) &= -2e^{-\mathcal{R}}(1+3\mathcal{R}^{-1}+3\mathcal{R}^{-2}) + 6\mathcal{R}^{-2}, \\ \mathcal{R} &= \lambda r. \end{aligned} \tag{1.6.9}$$

Since Brinkman equation and oscillatory Stokes equation are mathematically similar except for the meaning of λ, one may obtain the fundamental solution of Brinkman equation in a similar form.

1.7 Literature Review

Flow through porous media has received a considerable attention due to its wide range of applicability in various branches of science, engineering and industry. Engineering systems based on fluidized bed combustion, enhanced oil reservoir recovery, combustion in an inert porous matrix, groundwater flow, underground spreading of chemical waste and chemical catalytic reactors are just a few examples of applications of the study of flow through porous media. Another important application of flow through porous media in chemical engineering is 'chemical agglomeration' which is a frequently used process mainly in industry.

Several studies on Stokes flow past porous media present in literature assume that the porous media is in different forms. Macroscopic scale is the basis of continuum in porous media. The governing equation in the free flow region being steady state Stokes equations, Darcy's or Brinkman (generalized Darcy's) equation can be assumed in the region occupied by the porous matrix. In practical situations, porous particles may look like: a solid core covered with a porous layer, a composite porous slab, a porous shell or a porous cylinder / sphere and many other irregular forms. The problem of Stokes flow past spherical boundaries was initially introduced by Hasimoto (1956) where the problem of an axisymmetric flow past a rigid sphere is presented. Following this several studies came up on this topic (Collins, 1958; Shail, 1987). Happel and Brenner (1983) studied the steady flow of a Newtonian fluid past a solid sphere, liquid drop at low Reynolds number.

On the other hand, extensive studies have been done on viscous flow past porous spherical regions. The low Reynolds number flow past a porous spherical shell was studied by Jones (1973). It is assumed that Darcy's law governs the flow inside the porous spherical shell and the Beavers and Joseph (1967) boundary condition is used at the porous–liquid interface to solve the problem. The use of Darcy's law inside the porous region depends on the properties of porous media like permeability and porosity. For higher porosity Darcy's law is less valid and in such cases Brinkman equation is preferred. The problem of Stokes flow past porous particles using Brinkman equation started with the work of Higdon and Kojima (1981). Masliyah et al. (1987) studied the creeping flow over a porous spherical shell with a rigid core. They used Stokes equation and Brinkman equation for the flow in clear fluid and flow in porous region, respectively. Continuity of velocity, continuity of pressure and continuity of tangential stress at the interface of clear fluid and porous region are used. They obtained an analytical solution for a composite sphere and showed an excellent agreement with the experimental results. Yu and Kaloni (1988) obtained a cartesian tensor solution for the problem of Stokes flow past a Brinkman porous sphere and calculated the hydrodynamic force on the porous sphere in a uniform flow. Also, Qin and Kaloni (1993) have discussed the creeping flow past a porous spherical shell using Stokes and Brinkman equations. They obtained expressions for the velocities and pressure fields both inside and outside the porous spherical shell and determined the force acting on the shell. Adler (1981) studied the uniform flow and simple shear flow using Stokes equations in clear fluid and Brinkman equations inside a spherical aggregate. He obtained the streamlines around the porous particle and reported a critical value of dimensionless radius below which the particles are drained in case of simple shear flow. Palaniappan (1993) studied arbitrary Stokes flow past a porous sphere using Darcy's law inside the porous region and used the boundary conditions proposed by Jones (1973) at the porous–liquid interface and derived the corresponding Faxén's laws. Moreover, the limiting cases corresponding to rigid and shear–free spheres are also deduced. Raja Sekhar and Amaranath (1996) studied the problem of an arbitrary Stokes flow past a porous sphere with an impermeable core and Stokes flow inside a sphere with internal singularities enclosed by a porous spherical shell with an impermeable outer boundary. In both these problems, while matching Stokes equation with Darcy's law they have used Saffman (1971) condition for the tangential velocity components together with continuity of normal velocity and pressure. They derived corresponding Faxén's laws for drag and torque acting on the surface of porous spherical surface.

Padmavathi et al. (1993) suggested a representation for velocity and pressure fields of Brinkman equation while discussing a non–axisymmetric Stokes flow past a porous sphere. This representation expresses velocity and pressure in terms of scalars and the corresponding boundary conditions take a simple form in terms of these scalars. Moreover, this representation helps one to deal with arbitrary Stokes flow. Inspired by this Raja Sekhar et al. (1997) have shown that this representation of velocity and pressure is a complete general solution of Brinkman equation. The potential of this representation in order to deal with viscous flow problems involving spherical geometries is well established (see Palaniappan, 1993; Bhattacharyya and Raja Sekhar, 2004, 2005; Partha et al., 2005, 2006). Bhatt and Sacheti (1994) analyzed the problem of Jones (1973) by using Brinkman equation in place of Darcy's law inside the porous region. They have used continuity of velocity components, continuity of pressure, and continuity of tangential stress at the porous–liquid interface and calculated the drag force on the surface of the porous shell.

While several studies related to Stokes flow past impermeable or porous particles compute the drag force acting on the surface of the body, another important physical quantity is the torque acting on the surface of the body. A very efficient and compact way of representing drag and torque is in terms of Faxén's laws. It is well known in low Reynolds number hydrodynamics that Faxén's laws can be used to evaluate the drag force and torque on a solid impermeable particle subject to an arbitrary flow. A generalization of Faxén's laws was obtained by Keh and Chen (1996), which is applicable to the surface of an impermeable solid sphere with slip boundary condition. In case of liquid droplet, Hetsroni and Haber (1970); Hetsroni et al. (1971) obtained the corresponding Faxén's laws of force and torque, while Rallison (1978) derived the expression for the stresslet. Pozrikidis (1989) derived Faxén's laws corresponding to oscillatory Stokes flow past impermeable sphere using singularity method. Masliyah et al. (1987) solved the creeping flow of an incompressible Newtonian fluid past a spherically symmetric composite particle by using Stokes–Brinkman coupling. Majority of the above studies compute drag and torque on a porous particle when the flow is steady. In case of oscillatory Stokes flow past a porous sphere, there is no literature expressing drag and torque acting on the surface in terms of the corresponding Faxén's laws. In the next two chapters we derive these laws corresponding to a porous sphere in an arbitrary oscillatory flow. Such an investigation not only gives an idea of the hydrodynamic forces acting on the surface of a porous sphere, but also, the corresponding calculations can be used in order to understand the mass transfer inside porous pellets under oscillatory forcing. Another important application is to analyze acoustic properties of granular materials. Umnova et al. (2000) have considered oscillatory flow of viscous incompressible flow around a spherical particle and used cell model in order to estimate the hydrodynamic drag due to oscillating flow in a stack of fixed identical rigid spheres.

The literature survey on oscillatory Stokes flows inside porous media reveals that oscillatory forcing plays a significant role in understanding convective mass transfer in porous catalysts (Dragon and Grotberg, 1991; Dusting et al., 2006). This made us to take further interest in understanding the existing results related to this and make some significant contributions. In 'The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts', Aris (1975) presented a comprehensive review of the literature up to 1975. Since then, a number of scientific publications have considered the effect of transport by convection, in addition to diffusion, inside porous catalyst particles undergoing chemical reaction (Nir and Pismen, 1977; Rodrigues et al., 1984; Stephanopoulos and Tsiveriotis, 1989; Lu et al., 1993). In case of small, highly porous catalyst particles diffusion alone may not account for the nutrient transport and convective flow has a major role. It is evident that under steady state, convective flow within a porous catalyst is not so important whereas oscillatory forcing at higher amplitude and/or lower frequencies enhances the mass transfer. Ni et al. (2003) observed that oscillatory flow improves the performance of a bed packed with spherical particles. Crittenden et al. (2005) studied the influence of oscillatory flow on axial dispersion in packed beds of spheres. They observed that the best reduction (up to 50%) in the axial dispersion coefficient from the non–oscillation base value is at the highest frequency considered and when the column to particle size is the smallest. Also one may find importance of convective mass transport in various biological applications in (Prince et al., 1991; Frey et al., 1993; Ferguson et al., 2004).

The problems dealing with convection–diffusion–reaction can be solved in absence/

presence of external mass transfer at the boundary. In the field of biochemical engineering, oscillatory motion plays a significant role to enhance the mass transfer rate in biofilms and bioreactors (Nagaoka, 1997; Beeton et al., 1991). There are other examples where oscillatory motion has shown significant performance improvement such as ultrafiltration, microfiltration, electrodialysis, and electrophoresis (Perez-Herranz et al., 1997; Chandhok and Leighton, 1991; Najarian and Bellhouse, 1997; Blanpain-Avet et al., 1999). Stephanopoulos and Tsiveriotis (1989) studied the problem of convection and diffusion coupled with zero order isothermal reaction inside a spherical porous pellet. They have used the Dirichlet boundary condition at the pellet surface which can be achieved by neglecting external mass transfer. Dirichlet boundary condition has been used to deal with mass transfer inside various porous geometries (Rodrigues et al., 1992; Lu et al., 1993; Cardoso and Rodrigues, 2007). Peter and Böhm (2009) modeled a reaction–diffusion system in porous medium using the method of homogenization and discussed various scalings. While performing numerical experiments for simplified problems, they have tested the impact of Dirichlet and Robin type boundary conditions. There are several studies dealing with mass transfer by using Neumann boundary conditions (Miketinac et al., 1992; Wang and Brennan, 1995; Markowski, 1997). It is interesting to note that in case of steady flow, when the Dirichlet boundary condition is used together with a first order reaction kinetics, the internal concentration is a convex function lying between 0 and 1, while in case of zero order kinetics, the internal concentration becomes negative for some particular combination of parameters like Peclet number and Thiele modulus. Such a behavior is referred in literature as starvation zone. This physical ambiguity needs to be addressed and a reasonable criterion avoiding such starvation zones is required (see Stephanopoulos and Tsiveriotis, 1989). It is not known whether such a phenomena occur in case of oscillatory flows. Moreover, similar investigations using Robin type boundary condition will also give useful information in understanding physical situations involving convection, diffusion and reaction inside porous biological pellets. Chapters 4, 5 and 6 in the present thesis deal with the mass transfer inside porous pellets under oscillatory flow in absence/presence of external mass transfer.

In most of the studies where Stokes–Brinkman coupling is involved, continuity of velocity components together with either the continuity of all the stress components or the continuity of pressure and shear stress are widely accepted (see Levresse et al., 2001; Masliyah et al., 1987; Bhattacharyya and Raja Sekhar, 2004, 2005; Partha et al., 2005, 2006). However, Ochoa-Tapia and Whitaker (1995a,b) developed a momentum transfer condition at the boundary between a porous medium and a homogeneous fluid as a jump condition. This jump condition has been used by many researchers (Kuznetsov, 1996, 1998; Bhattacharyya and Raja Sekhar, 2004, 2005; Partha et al., 2005, 2006) to display the role of the jump coefficient. Bhattacharyya and Raja Sekhar (2004, 2005) have discussed viscous flow past a porous sphere with an impermeable core and Stokes flow inside a porous spherical shell. They have used stress jump boundary condition at the porous interface together with continuity of velocity components and normal stress. They derived the corresponding Faxén's laws for drag and torque acting on the surface of the porous sphere and the effect of stress jump coefficient is also shown when the basic flow is driven by either a Stokeslet or a Rotlet. Partha et al. (2005, 2006) have studied viscous flow past a porous spherical shell and viscous flow past a void in a porous bed respectively, using this stress jump boundary condition at the porus interface. Effect of jump coefficient on various quantities like volume flow, drag and torque is shown. In the present thesis, effect of this jump condition is discussed while estimating the overall bed permeability of beds of porous particle.

While every study related to porous media depends on permeability, the most important parameter characterizing the given porous media, measuring the overall bed permeability (OBP) of a given bed of particles is very useful. It is true that experimental methods (Rodrigues et al., 1995) and computational methods (Kim and Russel, 1985; Ladd, 1990; Kang and Sangani, 1994) give some estimates on measuring OBP, mathematical models have always been worth exploring to understand the situations at least in an approximate sense. A fluidized bed of porous particles may be considered as an assemblage of spherical/cylindrical porous particles. The interaction of neighboring particles may be taken into account via suitable boundary conditions. A representative porous particle surrounded by a fluid envelope may be considered in order to study various physical properties of a given bed. This model is known as cell model. Originally the cell model was portrayed as a rigid sphere surrounded by a fluid envelope (Happel, 1958) in assemblage of spherical particles. Happel (1958) proposed the cell boundary to be frictionless, which is also known as free surface cell model. Happel (1959) further extended the free surface cell model to the case of flow relative to cylinders and a good agreement with existing data on beds of fibres of various types was shown. There is another model where a porous particle in a fluid envelope is surrounded by another porous media. This model is known as effective medium model. Either the cell model or the effective medium model is a very useful tool to estimate the overall bed permeability (OBP) which is an important hydraulic parameter.

There are several studies present in literature which have dealt with theoretical prediction of OBP for the bed of porous particles via cell model and effective medium model (Davis and Stone, 1993; Li and Park, 1998, 2000; Albusairi and Hsu, 2004; Filippov et al., 2006; Raja Sekhar, 2010). Albusairi and Hsu (2004) have used effective medium model to study flow through beds of perfusive particles by using continuity of velocity, continuity of pressure and the continuity of tangential stress at the porous–liquid interface and obtained the overall perfusive bed permeability. In Albusairi's model the central representative particle is a porous sphere. Dodd et al. (1995) have successfully used the effective medium approximation to understand the hydrodynamic interaction on the diffusivities of integral protein membrane. Albusairi and Hsu (2005) have used effective medium model for a bed packed with perfusive particles to predict both diffusive term and the convective term in the correlation of the column axial dispersion coefficient.

All the above studies being analytical techniques, determine the OBP in an approximate sense, of the real physical models. Nevertheless, any such new analytical method in order to determine OBP gives more valuable information on the accuracy of various models. As a result of the recent discussions on the boundary conditions at a porous–liquid interface and the prospect of various boundary conditions on the fluid envelope in a cell model, we feel that estimating OBP for beds of porous spherical particles needs an attention. In the studies present in literature, Stokes equation is coupled with Brinkman equation and continuity of velocity components and continuity of stress components are used at the porous–liquid interface. Chapter 8 deals with the study of OBP using cell model and effective medium model. We have used the stress jump condition at the porous–liquid interface and given a theoretical prediction for OBP using both cell and effective medium model. A significant effect of jump coefficient on OBP is seen. A comparison with experimental data is also shown in both the cases.

It may be noted that majority of the analytical studies dealing with viscous flow through porous media considered either steady or unsteady Stokes equations which are linear and are coupled with Darcy / Brinkman equations for porous objects that are either spherical or cylindrical. At least in the steady case, it would be worth investigating a nonlinear system involving a porous–liquid interface. An obvious choice within the scope of the present study is to consider full Navier–Stokes equations together with Brinkman equation either for a spherical case or for a cylindrical case. Even if the problem of viscous incompressible flow at low Reynolds number past solid bodies is frequently studied by using the method of matched asymptotic expansions (MMAE), a few attention has been given to the corresponding problem in the case of porous bodies and in terms of the same procedure. Feng and Michaelides (1998) used a similar method to that developed by Proudman and Pearson (1957) for the problem of low Reynolds number viscous flow past a porous sphere, by assuming Darcy's law for the flow inside the porous sphere. However, recently, Kohr et al. (2008) have used the MMAE in order to study the two–dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape. Though the study done by Kohr et al. (2008) is developed in a more general case of Lipschitz domain and the corresponding problem in the case of a circular porous cylinder becomes a particular case of the above study, one gets explicitly the first two terms in the asymptotic expansion with respect to low Reynolds number of the force acting on the circular porous cylinder by using an alternative study based on the stream function formulation. Moreover, taking into account the fact that there are several choice of boundary conditions at a porous–liquid interface, the corresponding problem is worth investigating in the case of a circular porous cylinder.

1.8 Thesis Overview

This thesis consists of eight main chapters and a concluding chapter, i.e., Chapter 9 and a bibliography section. Chapter 1 is the introductory one whereas Chapters 2 to 8 deal with the work done in this thesis. Chapter 9 is the summary and conclusion of the work done in this thesis. At the beginning of each chapter, a brief survey of the literature is presented followed by an outline of the motivation in addressing the problem. The problem formulation is then presented with a detailed description of the solution procedure and its comparison with the existing literature. Results and discussions are given at the end of each chapter.

Chapter 1 is a global introduction of the entire thesis where all the necessary governing equations and boundary conditions are introduced, covering some relevant literature. The chapter starts with a brief description of porous media and transport phenomena is introduced. In this the momentum and mass transfer are discussed and the combined convection–diffusion–reaction equation is given. Then equations of motion start with equation of continuity and non–dimensionalization of Navier–Stokes equation is done. The linearized Stokes equations including the oscillatory Stokes equations are given. Since, the thesis deals with flow through porous pellets, the corresponding governing equations namely Darcy's law and Brinkman equation are introduced inside porous media. In case of three–dimensional flow, some complete general solutions of both Stokes and Brinkman equations are introduced. Prescribing appropriate boundary conditions is still a challenging topic for researchers and we have presented a systematic discussion on various possible boundary conditions at a porous–liquid interface together with merits and demerits in each case. Faxén's laws that are formulae corresponding to drag and torque acting on the surface of a spherical body are presented. Also, the fundamental solution of steady and oscillatory Stokes equations, called Stokeslet is introduced. Towards the end of this chapter, a detailed literature survey is given exposing the reader to important but a more general literature within the scope of the present study. It is important to mention here that the literature that is closely related to a particular problem discussed in individual chapters can be seen in the respective chapters.

The main aim of the next two chapters, i.e., **Chapter 2** and **Chapter 3** is to discuss the hydrodynamic problem of an arbitrary oscillatory Stokes flow past a stationary porous sphere in a viscous incompressible fluid. One may use Darcy's law or Brinkman equation inside the porous sphere. **Chapter 2** is the case of Darcy's law and **Chapter 3** is that of the Brinkman equation. In case of oscillatory Stokes–Darcy coupling, Saffman boundary condition for the tangential velocity together with continuity of normal velocity and continuity of pressure are employed. However, while matching oscillatory Stokes–Brinkman equations, continuity of velocity and stress is used. In each chapter, Faxén's laws that are compact expressions for drag and torque acting on the surface of a spherical body are derived. These formulae help one to compute the drag and torque acting on the surface of the porous sphere directly using the basic unperturbed flow. Some very useful examples like when the basic flow is driven by uniform oscillatory flow, oscillatory shear flow and oscillatory Stokeslet are discussed. Further, in each case the corresponding low–frequency limit is also discussed. These examples are helpful in understanding acoustic properties of granular materials or tissues.

In **Chapters 4** and **5**, nutrient transport inside a spherical porous pellet is studied. The flow inside the porous pellet is assumed to be governed by Darcy's law and that the flow outside the pellet by unsteady Stokes equation. The solution of the hydrodynamic problem of oscillatory Stokes flow past a porous sphere presented in **Chapter 2** has been used to formulate the convection–diffusion problem coupled with both zero and first order reaction kinetics in each case. Following the pseudo steady state approximation, the concentration has been assumed to be steady. In **Chapter 4**, the Dirichlet boundary condition, which can be achieved by neglecting the external mass transfer, is used at the surface of permeable porous pellet. Whereas **Chapter 5** takes into account the external mass transfer resistance leading to the Robin type boundary condition for the concentration. In each case, a series solution is obtained by computing the eigen functions corresponding to the elliptic operator. In case of zero order, it is observed that due to constant consumption, the nutrient gets exhausted at some points inside the pellet for a particular combination of parameters involved like Peclet number, Darcy number, frequency of oscillations, slip coefficient, etc. This is termed as starvation in literature. One witnesses such starvation zones corresponding to both Dirichlet condition (**Chapter 4**) and Robin condition (**Chapter 5**) when zero order reaction is considered. An optimality criterion is proposed to avoid such starvation zones. This optimality criterion gives a bound on the Thiele modulus as a function of Peclet number. A significant effect of oscillation on nutrient transport is seen. Based on this criterion, classification is done in order to identify the regions of nutrient sufficiency and starvation. However, in case of first order reaction, neither Dirichlet condition nor Robin condition forces such starvation for any combination of the parameters involved. The effect of oscillation on nutrient transport is analyzed in combination with the other parameters.

In **Chapter 6**, the problem of convection and diffusion coupled with isothermal zero and first order reaction, respectively is studied inside a cylindrical porous pellet. The flow inside the pellet is assumed to be governed by Darcy's law and that the flow outside the pellet by unsteady Stokes equation. The external mass transfer is neglected while discussing nutrient transport inside the pellet. The corresponding hydrodynamic problem is solved by using the stream function approach. The convection–diffusion–reaction problem is formulated and a Fourier series solution for the concentration inside the pellet is obtained. Similar to the spherical porus pellet, cylindrical pellet also experiences starvation for particular combination of parameters, of course with zero order reaction. This is not the case with first order reaction. A comparison between the spherical case and cylindrical case is also presented. It is observed that keeping all other parameters fixed, a cylindrical pellet experiences starvation at a smaller value of the Thiele modulus compared to a spherical pellet.

Chapter 7 deals with the method of matched asymptotic expansions (MMAE) in order to study the two–dimensional steady low Reynolds number flow of a viscous incompressible fluid past a porous circular cylinder. The flow inside the porous body is assumed to be described by the continuity and Brinkman equations and the velocity and stress fields are continuous across the interface between the fluid and porous media. Formal expansions for the corresponding stream functions are used, i.e., close to the body we have inner expansion called the Stokes expansion and in the outer domain we have outer expansion called the Oseen expansion. The matching principle has been employed at these two overlapping domains. It has been shown that the force exerted by the exterior flow on the porous cylinder admits an asymptotic expansion with respect to low Reynolds number, whose terms depend on the characteristics of the porous cylinder. In order to understand the matching procedure and the corresponding asymptotic solution for the coupled Navier–Stokes–Brinkman system, the case of a circular cylindrical pellet has been considered first. Following this, two other cases have also been discussed, the first one is the case of a circular porous cylinder assuming Darcy's law for the flow inside and the second is the case of a porous circular cylinder with a rigid core inside where Brinkman equation is considered for the flow in the porous region. Stress jump condition is used at the porous–liquid interface together with the continuity of velocity components and continuity of normal stress. The latter generates several particular cases like low Reynolds number flow past a solid circular cylinder. In each of the cases, we have shown the corresponding asymptotic expansion with respect to low Reynolds number, for the force that depends on the characteristics of the porous region.

Chapter 8 presents an analytical method to estimate an important hydraulic parameter, called the overall bed permeability (OBP). This is modeled as a problem of flow through beds of porous particles. In order to do this, two different models namely cell model and effective medium model are considered. Cell model assumes that a representative particle is inside a fluid envelope, on the surface of which the hydrodynamic interactions of the surrounding particles are to be prescribed in terms of suitable boundary conditions. On the other hand, effective medium model assumes that the inner representative particle experiences a porous medium in the vicinity, which is basically due to the surrounding particles. Stokes equation together with the equation of continuity is used in the clear fluid region, whereas porous region is assumed to be governed by Brinkman equation together with equation of continuity. Since the main idea is to incorporate the hydrodynamic interactions in some approximate sense, these are to be prescribed in terms of boundary conditions at the cell boundary. There are few popular boundary conditions such as, free surface boundary condition and the zero vorticity condition at the cell boundary. The free surface cell model has been extensively used to study both Newtonian and non–Newtonian fluid flow through cluster of bubbles, drops and permeable / impermeable particles. The free surface boundary condition and few other possible boundary conditions have been employed at the cell boundary. Though in case of Brinkman equation it is customary to use continuity of velocity components together with the continuity of stress components which are accepted by a large community, we have used the stress jump boundary condition at the porous–liquid interface. This is a more justified boundary condition at a porous–liquid interface whose validity is supported by recent literature. Several existing limiting cases are deduced and the variation of OBP with different parameters is analyzed. We have also made a comparison with experimental data point and seen a good agreement. Also comparison is made with the well known Carman–Kozeny model.

Finally, the summary and conclusions of the work done in Chapters 2 to 8 and future scope of the work are presented in **Chapter 9**.