Abstract

An $L(2,1)$-*coloring* (or *labeling*) of a graph G is a mapping $f : V(G) \to Z^{+} \cup \{0\}$ such that $|f(u)-f(v)| \geq 2$ for all edges uv of G, and $|f(u)-f(v)| \geq 1$ if u and v are at distance two in G. The *span of an* $L(2,1)$-*coloring* f of G, denoted by $\text{span}(f)$, is $\max\{f(v) : v \in V(G)\}$. The *span of* G, denoted by $\lambda(G)$, is the minimum span of all possible $L(2,1)$-colorings of G. If f is an $L(2,1)$-coloring of a graph G with span k then an integer l is a *hole* in f, if $l \in (0,k)$ and there is no vertex v in G such that $f(v) = l$. A *no-hole coloring* is defined to be an $L(2,1)$-coloring with no hole in it. An $L(2,1)$-coloring is said to be *irreducible* if the color of none of the vertices in the graph can be decreased and yield another $L(2,1)$-coloring of the same graph. An *irreducible no-hole coloring* of a graph G, in short *inh-coloring* of G, is an $L(2,1)$-coloring of G which is both irreducible and no-hole. A graph G is *inh-colorable* if there exists an inh-coloring of it. For an inh-colorable graph G the *lower inh-span* or simply *inh-span* of G, denoted by $\lambda_{inh}(G)$, is defined as $\lambda_{inh}(G) = \min\{\text{span}(f) : f \text{ is an inh-coloring of } G\}$. The *upper inh-span* of G, denoted by $\Lambda_{inh}(G)$, is defined as $\Lambda_{inh}(G) = \max\{\text{span}(f) : f \text{ is an inh-coloring of } G\}$.

In this thesis we first give answer to some problems asked in [Laskar, R., Jacob, J. and Lyle, J. (2010). Variations of graph coloring, domination and combinations of both: a brief survey, *Advances in Discrete Mathematics and Applications; Ramanujan Mathematical Society Lecture Notes Series.* 13, 133-152] These problems are mainly about relationship between the span and maximum no-hole span of a graph, lower inh-span and upper inh-span of a graph, and the maximum number of holes and minimum number of holes in a span coloring of a graph. We also give some sufficient conditions for a tree and an unicyclic graph to have inh-span $\Delta + 1$. We prove that the hypercube Q_n is inh-colorable for every $n \geq 4$ and find upper bounds of its inh-span and upper inh-span. We find the exact value of the inh-span of both finite and infinite triangular lattices.

Then we work on $L(2,1)$-coloring and inh-coloring of $K_m \square C_n$ and find the exact value of $\lambda(K_m \square C_n)$ and $\lambda_{inh}(K_m \square C_n)$ for some m and n, and give upper bounds in the remaining cases. We prove that for $n \geq 3$, the Cartesian product of a tree and a path P_n is inh-colorable and give an upper bound to the inh-span of the same. We also give two upper bounds to the span of Cartesian product of two arbitrary graphs which are better than the existing known upper bound in some cases. We prove that for $m \geq 4, n \geq 3$, the direct product $K_m \times P_n$ of complete graph and path is inh-colorable and find the exact value of its inh-span except one case. We determine the exact value of $\lambda(K_m \times C_n)$ when $m(\geq 3)$ is odd and n is a multiple of $5m+1$, and give an improved upper bound of $\lambda(K_m \times C_n)$ for some other values of m and n. We prove that for $m \geq 3, n \geq 4$, $K_m \times C_n$ is inh-colorable and find the exact value of $\lambda_{inh}(K_m \times C_n)$ for $n = 4, 5$ or 6 and an upper bound of it for the remaining values of n. For $n \geq 6$ and except finitely many values of m we show that $P_n \times C_m$ is inh-colorable and find the exact value of $\lambda_{inh}(P_n \times C_m)$. We also give two upper bounds to the span of direct product of two arbitrary graphs. We give improved upper bounds to the span of lexicographic product of arbitrary graphs. We study inh-colorability of $P_n \circ G, C_n \circ G, C_n \circ \overline{K_m}$ and $T \circ \overline{K_m}$, where G is an arbitrary graph and T is a tree different from a star and a path. We show that for an arbitrary graph G_1, if $G_1 \circ \overline{K_m}$ is inh-colorable then for any graph G_2 on m vertices with $\lambda(G_2) \leq m-1$, $G_1 \circ G_2$ is inh-colorable and $\lambda_{inh}(G_1 \circ G_2) \leq \lambda_{inh}(G_1 \circ \overline{K_m})$.

We show that the subdivision graph $G_{(h)}$ is inh-colorable for $h(e) \geq 2$, $e \in E(G)$. Moreover we find the exact value of $\lambda_{inh}(G_{(h)})$ in several cases and give upper bounds of the same in the remaining. We prove that the edge-multiplicity-paths-replacement graph $G(rP_h)$ is inh-colorable except possibly the following cases: $h(e) = 2$ for at least one but not for all e and $\Delta(G) = 2$, $r = 2$ or $\Delta(G) \geq 3$, $2 \leq r \leq 4$. We find the exact value of $\lambda_{inh}(G(rP_h))$ in several cases and give upper bounds to the same in the remaining. Moreover we find the exact value of $\lambda(G(rP_h))$ in most of the cases which were left by Lü and Sun [Lü, D. and Sun, J. (2016). L(2, 1)-labelings of the edge-multiplicity-paths-replacement of a graph, *Journal of Combinatorial Optimization*. 31, 396-404.].
Keywords: $L(2,1)$-coloring; No-hole coloring; Irreducible coloring; Irreducible no-hole coloring; Span of a graph; Unicyclic graph; Hypercube; Triangular lattice; Cartesian product of graphs; Direct product of graphs; Lexicographic product of graphs; Subdivision graph; Edge-multiplicity-paths-replacement Graph.