1 Introduction

1.1 Motivation

The *discrete time nonlinear dynamical systems* or *maps* are interesting in their own right, as mathematical laboratories for chaos. The Poincaré mapping technique for flows condenses the behavior of n-dimensional trajectories to a mapping of a $(n-1)$-dimensional surface of section to itself. The dynamical properties of the map reflect the dynamical properties of the flow. In some scientific disciplines it is natural to sample the states at discrete intervals of time, for example, power electronics, parts of economics and finance theory, impulsively driven mechanical systems, and some specific animal population models where successive generations do not overlap (Strogatz (1994); Glendinning (1999); Hilborn (2000); Thompson and Stewart (2001); Sprott (2003)).

Most of the past studies on dynamical systems focused mainly on *smooth functions* (Kuznetsov (1998); Hilborn (2000)). This approach proved very useful in understanding the dynamics of many important physical phenomena such as fluid flows, population dynamics, etc. However, the theory of smooth dynamical systems leaves many physically significant systems beyond its purview. These are known as *nonsmooth* or *piecewise smooth* dynamical systems involving nonsmooth system functions (Banerjee and Vergh-

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ese (2001); Zhusubaliyev and Mosekilde (2003); Leine and Nijmeijer (2004); di Bernardo et al. (2007); Simpson (2010)). In a piecewise smooth dynamical system, the phase space is partitioned into many finite regions in each of which the system is sufficiently smooth, and the boundary between different regions is called a *switching manifold*, or sometimes, a *border*. Important examples of piecewise smooth systems are: electrical switching circuits (Banerjee et al. (1998); Banerjee and Grebogi (1999); Banerjee et al. (2000a,b)), impacting systems (Nordmark (1991); Fredriksson and Nordmark (1997); Ing et al. (2008)), stick-slip oscillations (Dankowicz and Nordmark (2000)), cardiac dynamics (Sun et al. (1995); Hassouneh and Abed (2004); Berger et al. (2007); Zhao et al. (2008)), walking robots (Thuilot et al. (1997)), economic systems (Agliari et al. (2006)) and biological systems (Dercole (2005); Dercole et al. (2007); Simpson et al. (2009); Polynikis et al. (2009); Durrett (2010)), etc.

A qualitative change in the dynamics of a system is called *bifurcation*. There are two main types of bifurcations – local (e.g., *saddle-node, period-doubling, Neimark-Sacker*) and global (e.g., *homoclinic and heteroclinic*). One can define local bifurcations in discrete-time systems as those detectable in any small neighborhood of a fixed point (or periodic orbit). In other words, a local bifurcation in a map occurs when the change of a parameter changes the stability of a fixed point (or a periodic orbit). There are also bifurcations that cannot be detected by only looking at a small neighborhood of fixed points. Such bifurcations are called global. Global bifurcations occur when *larger* invariant sets, such as chaotic orbits, stable and unstable manifolds, etc., collide with each other or with periodic orbits. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood. In fact, the changes in topology extend out to a large distance (hence *global*). It is known that piecewise smooth maps display many bifurcations which do not occur in smooth maps. Nonsmooth or so-called *border collision bifurcation* occurs in piecewise smooth maps

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when a fixed point (or a periodic orbit) collides with the border, resulting in an abrupt change in the Jacobian matrix (Nusse and Yorke (1992); Banerjee and Verghese (2001); Zhusubaliyev and Mosekilde (2003); di Bernardo et al. (2007)) evaluated at the fixed point (or the periodic orbit). Nowadays, the bifurcation theory of smooth maps is quite well developed (Guckenheimer and Holmes (1983); Kuznetsov (1998)) although for piecewise smooth maps it is still far from being complete. In order to explain and control the behavior of nonsmooth maps, it is necessary to develop both local and global bifurcation theory for such maps. Therefore, in this thesis, one of our goals is to enrich the basic principle of bifurcation phenomena of the nonsmooth maps.

Most of the studies on border collision bifurcation have been conducted by considering the variation of only one parameter—which is most relevant to the bifurcation under study—keeping the other parameters fixed (Nusse and Yorke (1992); Nusse et al. (1994); Nusse and Yorke (1995); Banerjee and Grebogi (1999); Banerjee et al. (2000a); Jain and Banerjee (2003)). However, some recent works (Avrutin and Schanz (2006); Avrutin et al. (2006, 2007b, 2010a)) described the characteristics of one-dimensional piecewise smooth maps in the whole parameter space from the point of view of multiparameter bifurcation study. These studies revealed certain structures in the parameter space, which explain the typical nonsmooth bifurcation sequences like period-adding cascade, period-incrementing cascade, etc. Logically, the next step was to extend this methodology to two-dimensional piecewise smooth maps. This line of work is fraught with the problem that the calculation of the regions of existence of high periodic orbits in the parameter space often leads to very complicated equations that cannot be handled easily. In this thesis we have solved this problem using an approach first proposed by Leonov (Leonov (1959, 1960a,b)), and have thereby elaborately calculated the regions of existence and stability of various high-periodic orbits in the parameter space.

In the study of dynamical systems, much progress has been made in understanding

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the transition to chaos through torus breakdown. Together with the local bifurcations, torus destruction represents one of the classical routes to chaos. In a seminal paper, Afraimovich and Shilnikov (1991) outlined three fundamental routes to chaos through torus breakdown in smooth systems. However, for nonsmooth systems the interaction between the border and invariant set can lead to qualitatively different mechanisms of transition to chaos. This thesis also investigates the mode-locked torus destruction routes to chaos in two- and three-dimensional piecewise smooth maps.

1.2 An Overview on Piecewise Smooth Discrete Dynamical Systems

A dynamical system is defined as a deterministic mathematical prescription whose state evolves forward in time (Ott (1993)). A *nonsmooth* or *piecewise smooth* dynamical system $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is everywhere smooth, i.e., $\mathcal{C}^n$ for some $n \in \mathbb{N}$, except on some boundaries $\mathcal{S}_m$ called *switching manifolds* or *borders*, which divide the phase space $\mathcal{D} \subset \mathbb{R}^n$ into countably many regions (Banerjee and Verghese (2001)). In a nonsmooth dynamical system, time can be treated as a continuous variable or as a discrete integer-valued variable. A continuous time evolution is usually described by a system of nonsmooth differential equations, while a discrete time evolution is described by a system of difference equations (known as nonsmooth map). In this dissertation, we have mainly investigated the dynamics of nonsmooth maps which contain just one discontinuity boundary. That is, by an appropriate choice of local co-ordinates, the maps under investigation can be described as:

$$\tilde{x} \mapsto \mathcal{G}(\tilde{x}, \tilde{\rho}) = \begin{cases} \mathcal{G}_1(\tilde{x}, \tilde{\rho}) & \text{if } \mathcal{H}(\tilde{x}, \tilde{\rho}) \leq 0, \\ \mathcal{G}_2(\tilde{x}, \tilde{\rho}) & \text{if } \mathcal{H}(\tilde{x}, \tilde{\rho}) > 0, \end{cases} \tag{1.1}$$

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where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ...\tilde{x}_n) \in \mathbb{R}^n$, $\tilde{\rho} \in \mathbb{R}$, $\mathcal{G}_{1,2} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth functions of $\tilde{x}$. The switching manifold is given by

$$\mathcal{S}_m := \{\tilde{x} \in \mathbb{R}^n : \mathcal{H}(\tilde{x}, \tilde{\rho}) = 0\},$$

which separates $\mathbb{R}^n$ into two regions $\mathcal{R}_1 := \{\tilde{x} \in \mathbb{R}^n : \mathcal{H}(\tilde{x}, \tilde{\rho}) < 0\}$ and $\mathcal{R}_2 := \{\tilde{x} \in \mathbb{R}^n : \mathcal{H}(\tilde{x}, \tilde{\rho}) > 0\}$. Now we intend to study the dynamics of the map (1.1) caused by the movement of a fixed point (or a periodic orbit) across the border. This can be done by considering the local linear approximation in a sufficiently small neighborhood of the border collision point in both state space and parameter space.

Using appropriate series expansion and co-ordinate transformation, the local linear approximation of the map (1.1) in the neighborhood of the border collision fixed point at $x = 0,\ \rho = 0$, can be expressed as:

$$x \mapsto \mathcal{F}(x, \rho) = \begin{cases} \mathcal{F}_1(x, \rho) = \mathcal{A}_1 x + \mathcal{C}_1 \rho & \text{if } \mathcal{R}_1 = \mathcal{C}^T x \leq 0, \\ \mathcal{F}_2(x, \rho) = \mathcal{A}_2 x + \mathcal{C}_2 \rho & \text{if } \mathcal{R}_2 = \mathcal{C}^T x > 0, \end{cases} \tag{1.2}$$

where

$$\mathcal{A}_1 = \left.\frac{\partial \mathcal{G}_1}{\partial x}\right|_{x=0,\ \rho=0},\quad \mathcal{A}_2 = \left.\frac{\partial \mathcal{G}_2}{\partial x}\right|_{x=0,\ \rho=0},\quad \mathcal{C}_1 = \left.\frac{\partial \mathcal{G}_1}{\partial \rho}\right|_{x=0,\ \rho=0},\quad \mathcal{C}_2 = \left.\frac{\partial \mathcal{G}_2}{\partial \rho}\right|_{x=0,\ \rho=0},$$

$\mathcal{A}_1 = [a_{ij}^{(1)}]$ and $\mathcal{A}_2 = [a_{ij}^{(2)}]$ are real $n \times n$ matrices, $\mathcal{C}_k = (c_{i1}, c_{i2}, \ldots, c_{in})^T,\ k = 1, 2$; is an $n \times 1$ matrix, $\mathcal{C}^T = (1, 0, \ldots, 0)^T$ is an $n \times 1$ matrix, $(a_{ij}^{(1)},\ a_{ij}^{(2)},\ c_{ij}) \in \mathbb{R}$, $i, j = 1, 2, \ldots, n$ and the switching manifold is given by $\mathcal{S}_m := \{x \in \mathbb{R}^n : \mathcal{C}^T x = 0\}$. One can apply a suitable coordinate transformation to obtain the matrices in the normal form, where the matrices $\mathcal{A}_1 = [a_{ij}^{(1)}]$ and $\mathcal{A}_2 = [a_{ij}^{(2)}]$ are such that $a_{ij}^{(1)} = a_{ij}^{(2)}$, $\forall j \neq 1$, i.e., except the

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first columns of the matrices all other columns are same*. In this thesis we have studied normal form piecewise smooth maps of the form (1.2). Below we give some important definitions which are necessary for our dynamical inventory.

1.2.1 Definitions

Definition 1.2.1. *A point $x = x^*$ is said to be an* **admissible fixed point** *of the map* (1.2) *if, $\mathcal{F}_i(x^*, \rho) = x^*$ for $i = 1, 2$ and $x^* \in \mathcal{R}_j$ with $j = 1, 2$ and $j = i$.*

Definition 1.2.2. *A point $x = \tilde{x}^*$ is said to be a* **virtual fixed point** *of the map* (1.2) *if, $\mathcal{F}_i(\tilde{x}^*, \rho) = \tilde{x}^*$ for $i = 1, 2$ and $\tilde{x}^* \in \mathcal{R}_j$ with $j = 1, 2$ and $j \neq i$.*

Admissible fixed points have same dynamics like the fixed points in smooth dynamical systems, but virtual fixed points exist only for the piecewise smooth systems. Moreover, the virtual fixed points can still attract or repel iterates.

Definition 1.2.3. *A* **border collision bifurcation** *of a fixed-point x^* (or a periodic orbit) for the map* (1.2) *is defined as the event when x^* (or the periodic orbit) transversely crosses the switching manifold $\mathcal{S}_m := \{x \in \mathbb{R}^n : \mathcal{C}^T x = 0\}$ and the eigenvalues of x^* (or the periodic orbit) abruptly jump from the inside of the unit circle to the outside it or vice versa.*

Definition 1.2.4. *A border collision bifurcation is called* **persistence** *if an admissible fixed point and a virtual fixed point of the map* (1.2) *hit the boundary $\mathcal{S}_m$ and then the admissible fixed point turns into a virtual fixed and the virtual fixed point turns into an admissible fixed point.*

There exist three basic types of local border collision or nonsmooth bifurcations (see Figure 1.1): (i) *nonsmooth saddle-node*, (ii) *nonsmooth period-doubling*, and (iii) *non-*

*The derivation of the normal form map in one-dimension can be found in Banerjee et al. (2000a), that in two-dimension can be found in Yuan (1997) and Banerjee and Grebogi (1999), and the same in three-dimension can be found in Roy and Roy (2008).

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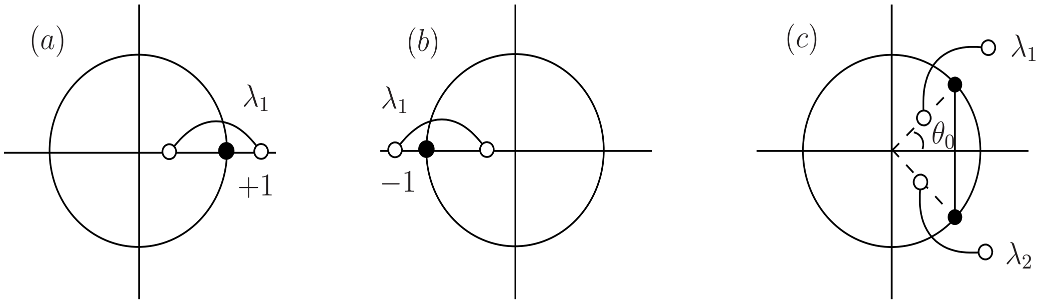

Figure 1.1: The discontinuous jumps of the eigenvalues in piecewise smooth maps resulting in (a) nonsmooth saddle-node bifurcation, (b) nonsmooth period-doubling bifurcation, and (c)nonsmooth Neimark-Sacker.

smooth Neimark-Sacker bifurcation. These are the nonsmooth analogues of classical smooth bifurcations.

Definition 1.2.5. Nonsmooth saddle-node bifurcation: *A bifurcation is called nonsmooth saddle-node bifurcation when two co-existing admissible fixed points* $x_1^* \in \mathcal{R}_1$ *and* $x_2^* \in \mathcal{R}_2$ *of the map* (1.2) *cross the switching manifold* $\mathcal{S}_m$ *and change continuously into two virtual fixed points* $\tilde{x}_1^* \in \mathcal{R}_2$ *and* $\tilde{x}_2^* \in \mathcal{R}_1$*, respectively.*

Definition 1.2.6. Nonsmooth period-doubling bifurcation: *A bifurcation is called nonsmooth period-doubling when a stable fixed point* x^* *collides with the border* $\mathcal{S}_m$ *and becomes unstable and a period-2 orbit* (x_1^*, x_2^*) *is created having one iteration on each side of the border.*

Definition 1.2.7. Nonsmooth Neimark-Sacker bifurcation: *In this bifurcation a fixed point* x^* *abruptly changes its stability via a pair of complex eigenvalues with unit modulus.*

The Neimark-Sacker bifurcation can be supercritical or subcritical, resulting in a stable or unstable (within an invariant two-dimensional manifold) closed invariant curve, respectively. When it happens in the Poincaré map of a limit cycle, the bifurcation generates an invariant two-dimensional torus in the corresponding ordinary differential equation.

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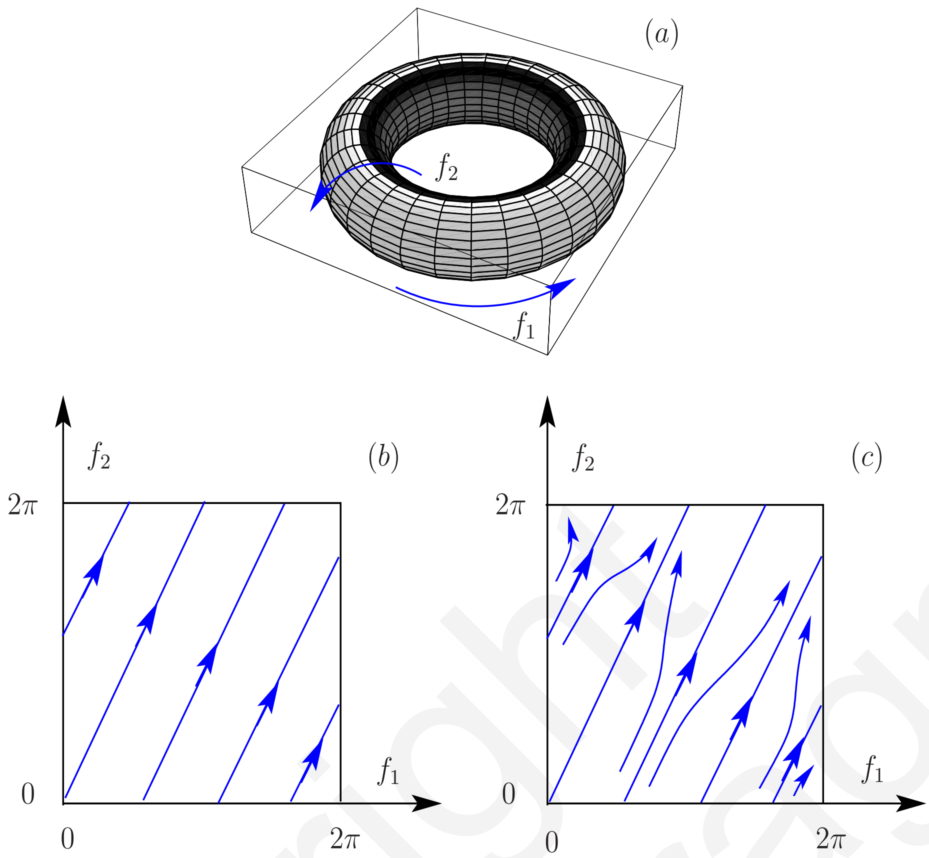

Figure 1.2: (a) A torus associated with 2-frequencies f_1 and f_2. Phase portraits: Perturbation from (b) a quasiperiodic to (b) a mode-locked 2-frequency torus with suitable co-ordinates f_1, f_2 modulo 2π (Broer et al. (1996)).

Definition 1.2.8. *An* **invariant torus** *is associated with a motion in the phase space with a finite number of frequencies, usually two (say f_1 and f_2). Dynamics on a torus may be* **quasiperiodic** *when it contains no periodic orbits, or it may be* **mode-locked** *into containing a stable and an unstable periodic orbit, which wind a given number of times around the torus. For a quasiperiodic motion the ratio of the two frequencies f_1/f_2 is an irrational number, i.e., the two frequencies are incommensurate, and if the ratio of f_1/f_2 is a rational number, i.e., they are commensurate, it is called a mode-locked periodic motion (see Figure 1.2).*

Definition 1.2.9. *Without going into mathematical details one may define* chaos *as an*

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effectively unpredictable long term behavior arising in a deterministic dynamical system because of sensitivity to initial conditions. It must be emphasized that a deterministic dynamical system is perfectly predictable given perfect knowledge of the initial conditions, and is in practice always predictable in the short term. The key to long-term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions. No matter how precisely the initial condition in these systems is measured, prediction of its subsequent motion goes radically wrong after a short time. Typically the predictability grows only logarithmically with the precision of measurement.

Definition 1.2.10. *Let x^* be a fixed point (or a point of a periodic orbit) for the map $\mathcal{F}$, the local* **stable manifold** *of x^* is the set of all points defined as:*

$$W^s_{loc}(\mathcal{F}, x^*) = \{y \in \mathbb{R}^n : \mathcal{F}^n(y) \to x^* \ \ as \ \ n \to \infty\}$$

and the local **unstable manifold** *of x^* is defined by:*

$$W^u_{loc}(\mathcal{F}, x^*) = \{y \in \mathbb{R}^n : \mathcal{F}^{-n}(y) \to x^* \ \ as \ \ n \to \infty\}.$$

Definition 1.2.11. *Let, $\mathcal{F}(x) = x$ and $\mathcal{F}'(x) > 1$, a point y is called* **homoclinic** *to x if $y \in W^u_{loc}(\mathcal{F}, x)$ and there exists $n > 0$ such that $\mathcal{F}^n(y) = x$. The point y is* **heteroclinic** *if $y \in W^u_{loc}(\mathcal{F}, x)$ and there exists $n > 0$ such that $\mathcal{F}^n(y)$ lies on a different periodic orbit. A* **homoclinic orbit** *is a trajectory x_n, $n \in \mathbb{N}$ that connects an equilibrium x to itself, i.e., $x_n \to x$ as $n \to \pm\infty$. A* **heteroclinic orbit** *connects two different equilibria x and y, i.e., $x_n \to x$ as $n \to +\infty$ and $x_n \to y$ as $n \to -\infty$.*

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1.2.2 Example of a piecewise smooth discrete dynamical system

In this thesis, we have mainly investigated the dynamics of piecewise smooth maps, specifically various normal form piecewise smooth maps. That is why, in illustrating the dynamical phenomena to be investigated in this thesis, it will be inappropriate to take the usual route of smooth maps like the logistic map or the Hénon map. For this purpose we have considered the *Lozi map*, which is a two-dimensional piecewise linear map (Lozi (1978)). The mapping equation can be written as:

$$\mathcal{L}_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 - a|x|^{\gamma} + y \\ bx \end{pmatrix} \text{ with } \gamma = 1. \tag{1.3}$$

Here $a, b \in \mathbb{R}$ are system parameters, different values of which will lead to different system behavior. The switching manifold $\mathcal{S}_m$ is given by the x-axis. The state space can be divided into two compartments $\mathcal{L} := \{(x, y) : x < 0, y \in \mathbb{R}\}$ and $\mathcal{R} := \{(x, y) : x > 0, y \in \mathbb{R}\}$. Simple algebraic calculation shows that the map (1.3) has two fixed points given by

$$\mathcal{O}_{\mathcal{L}} = \left(\frac{1}{1 - a - b}, \frac{b}{1 - a - b} \right),$$

in the region $\mathcal{L}$ and

$$\mathcal{O}_{\mathcal{R}} = \left(\frac{1}{1 + a - b}, \frac{b}{1 + a - b} \right),$$

in the region $\mathcal{R}$. The fixed point $\mathcal{O}_{\mathcal{L}}$ exists if $b > 1 - a$ and the fixed point $\mathcal{O}_{\mathcal{R}}$ exists if $b < 1 + a$. Moreover, one can easily determine the local stability of these points by calculating the eigenvalues of the Jacobian matrix

$$J_{\mathcal{L}_{a,b}} = \begin{pmatrix} -a\frac{\partial |x|^{\gamma}}{\partial x} & 1 \\ b & 0 \end{pmatrix}$$

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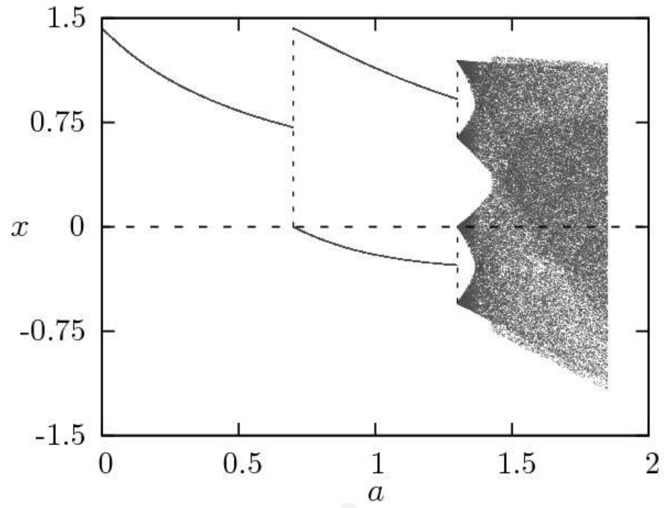

Figure 1.3: Bifurcation diagram of the Lozi map given by (1.3) for $b = 0.3$. Dashed vertical lines show a continuum of neutral period-2 and period-4 orbits respectively, coexisting at the border collision bifurcation point.

evaluated at the fixed points. The determinant of the Jacobian matrix is constant; $J_{\mathcal{L}_{a,b}} = -b$. The case of $|b| < 1$ corresponds to the dynamics of a dissipative system, and that of $|b| = -1$ to a conservative (Hamiltonian) system. In the limit $b \to 0$ the map becomes one dimensional. The map is invertible for $b \neq 0$, and is invariant under the transformation $\mathcal{L}_{a,b}(x, y) \to \mathcal{L}_{a,b}(-x, -y)$.

Depending on the values of the system parameters a and b, the map demonstrates a range of dynamical behavior from predictable to chaotic. Figure 1.3 shows the bifurcation diagram of the map (1.3). In this diagram the values of x, after eliminating the transients, have been plotted for a range of values of the parameter a with $b = 0.3$. Now we briefly look at the dynamics of the map.

The route to chaos for the piecewise smooth maps can be very different from that in smooth maps. For example, in the Lozi map, no usual period-doubling route to chaos is possible. This is due to the lack of continuity in the derivative of the map. For the map (1.3), with the variation of the parameter a, first there exists a stable period-1 orbit $\mathcal{O}_{\mathcal{R}}$. The fixed point $\mathcal{O}_{\mathcal{L}}$ does not exist for $a \leq 0.7$ and is unstable everywhere. At $a \approx 0.7$,

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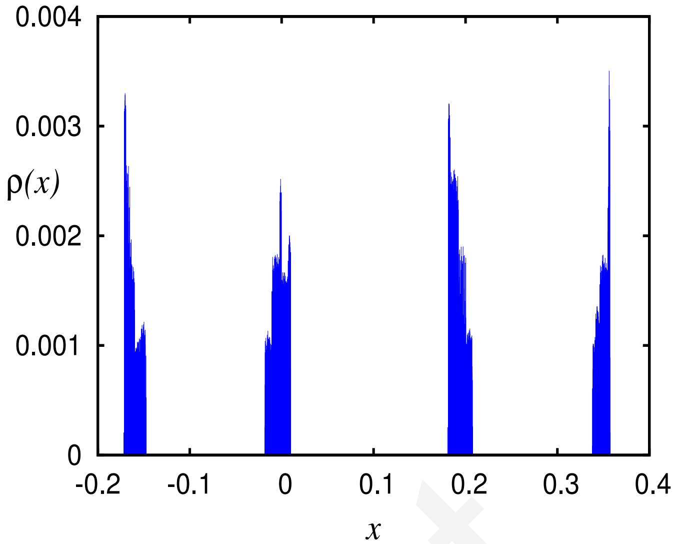

Figure 1.4: Invariant measure of a four-band chaotic attractor at $a = 1.32$ and $b = 0.3$ of the Lozi map (1.3).

an eigenvalue of the period-1 orbit $\mathcal{O}_{\mathcal{R}}$ crosses the unit circle through -1 and a period-doubling bifurcation occurs. However, this period-doubling bifurcation is quite different from the smooth period-doubling, at this bifurcation point two neutral period-2 orbits are born and one of them hits the border and then both of them disappear. At this point a stable period-2 orbit is born, it goes to a similar bifurcation at $a \approx 1.3$ and a four-band chaotic attractor appears via a border collision bifurcation. The invariant measure of the four-band chaotic attractor is depicted in Figure 1.4. With further variation of the parameter the four-band chaotic attractor first becomes a two-band and then one-band chaotic attractor. Finally the one-band chaotic attractor disappears through a boundary crisis.

Our numerical investigations of the Lozi map $\mathcal{L}_{a,b}$ for the values of the parameters $a = 1.78$ and $b = 0.3$ (see Figure 1.5(a)) suggest the existence of strange attractor. In Figure 1.5(b) we have presented the strange attractor for the values of parameters $a = 1.4$ and $b = 0.3$ with $\gamma = 2$. It is known that for $\gamma = 2$ the map (1.3) becomes the well known Hénon map (Hénon (1976)), which is a smooth map. One can easily see

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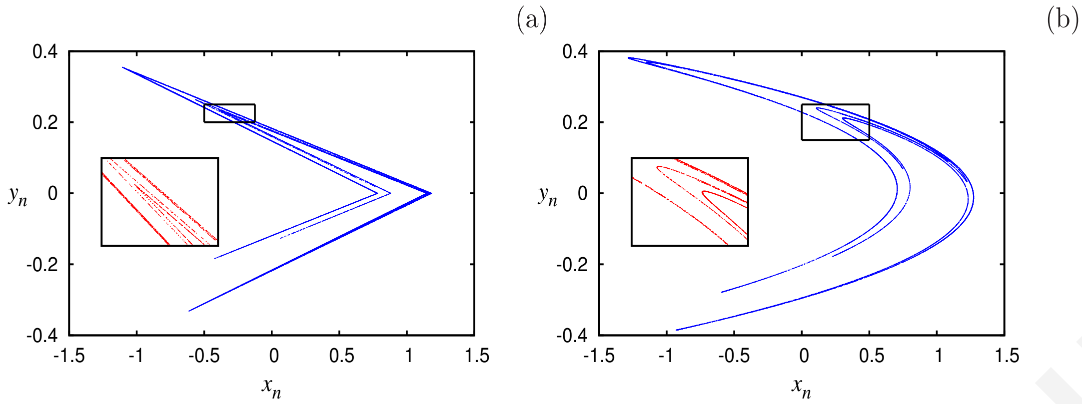

Figure 1.5: Chaotic attractor of the map (1.3): (a) for $\gamma = 1$ (the Lozi map) with $a = 1.78$ and $b = 0.3$, and (a) for $\gamma = 2$ (the Hénon map) with $a = 1.4$ and $b = 0.3$. The portion of the attractors marked by the rectangles are shown in the inside sub-figures.

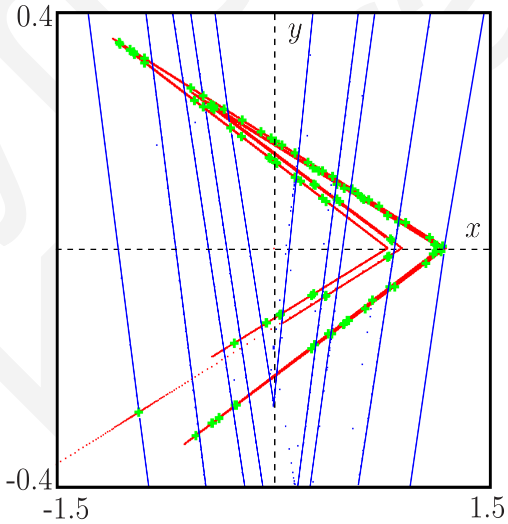

Figure 1.6: The chaotic attractor of the Lozi map for $a = 1.78$ and $b = 0.3$, formed by the unstable manifolds of the chaotic saddles. The stable manifolds are marked by blue lines, the unstable manifolds are marked by red lines and the chaotic saddles are marked with green crosses.

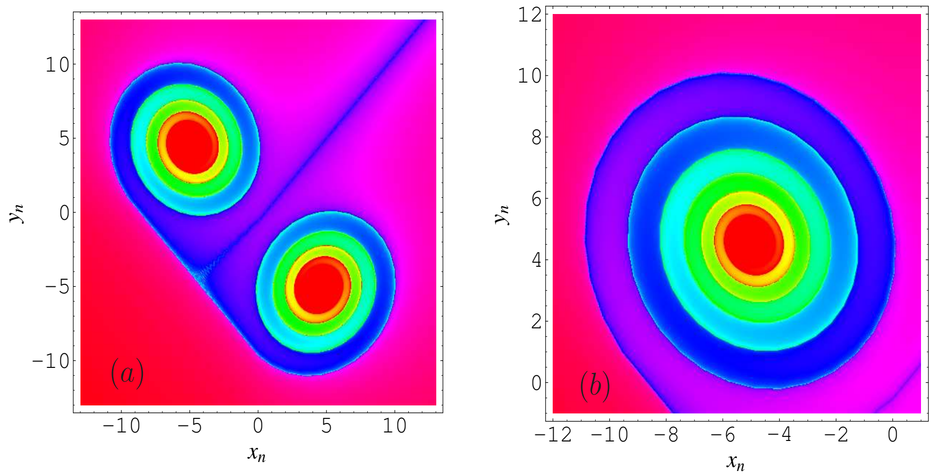

Figure 1.7: Escape time plot of the Lozi map (1.3) for $a = 0.21$ and $b = 1.01$. In (a) two regions of space for the map colored according to the number of iterations required to escape, and (b) a zoomed region of (a). In the numerical simulation we have used Maximal iteration number: 450 and explosion threshold value: $x^2 + y^2 = 450$. Points for which these values reach the threshold were plotted in red, others in color depending on the number of iterations reached until the explosion value was exceeded (Michelitsch and Rössler (1989)).

how the structure of the unstable manifolds (which forms the strange attractor) changes from nonsmooth to smooth by changing $\gamma = 1$ to 2. The stable and unstable manifolds of the chaotic saddles forming the strange attractor are drawn in Figure 1.6. Owing to the piecewise nature of the map the manifolds are formed by the line segments, and the unstable manifolds fold along the x-axis and the stable manifolds fold along the y-axis.

Using the technique (Michelitsch and Rössler (1989)) of plotting the escape times from repellers the picture of Figure 1.7(a) is obtained for specific values of the parameters a and b. Figure 1.7(b) shows a blow up of the left-hand eye. Actually for the Lozi map, close to its conservative limit, we study trajectories which tend towards infinity for $n \to \pm\infty$, and which are trapped in the vicinity of the invariant set for long times.

This simple example of a piecewise map shows the occurrence of bifurcations leading to transition from a period-1 attractor to a period-2 attractor without usual period-

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doubling, direct transition from a periodic attractor to a chaotic attractor, etc. These behaviors are very specific to piecewise smooth dynamical systems. Other behaviors of normal form piecewise smooth maps will be discussed later in this thesis.

1.3 Literature Review

In the early seventies, the Russian mathematician Mark I. Feigin seems to be among the first to describe border collision bifurcation, which he called "C-bifurcation"[†] (Feigin (1970, 1974)). Feigin outlined an analytical framework for the existence of fixed point and periodic solutions for general n-dimensional piecewise smooth maps. Some of the main results of his work have subsequently been recast into the framework of modern bifurcation theory by di Bernardo et al. (1999). The term *border collision bifurcation* was coined in the pioneering work by Nusse and Yorke (1992). They had obtained some invaluable results on piecewise smooth maps (Nusse and Yorke (1992, 1995); Nusse et al. (1994)). Among other phenomena, they emphasized the possibility of observing peculiar bifurcations such as direct transitions from period-2 to period-3 and regular periodic motion to chaos. Border collision bifurcations include bifurcations that are reminiscent to that of saddle-node (fold), period-doubling (flip) and Neimark-Sacker bifurcations in smooth systems. However, there are many atypical bifurcations (including the degenerate cases (Avrutin et al. (2010c); Sushko and Gardini (2010))), that have no smooth counterpart.

Nowadays, the smooth dynamical systems theory has a firm footing but the piecewise smooth or nonsmooth dynamical systems theory is still far from being complete, and certainly very preliminary in comparison to the results available in the smooth case

[†] "C" stands for the Russian word for "sewing". This refers to the fact that one has to "sew" the solutions together across the border between two smooth regions.

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(Banerjee and Verghese (2001); Zhusubaliyev and Mosekilde (2003); Leine and Nijmeijer (2004); di Bernardo et al. (2007); Simpson (2010)). A complete classification of possible border collision bifurcations is only available for one-dimensional piecewise smooth maps (di Bernardo et al. (1999); Banerjee et al. (2000a)). In order to explain and control the bifurcation phenomena in such systems it is necessary to develop the border collision bifurcation theory for higher dimensional piecewise smooth maps. It is known that for smooth systems, the concept of dimension reduction via "center manifold theorem" allows bifurcations in systems of arbitrary dimensions to be studied in terms of their low-dimensional normal forms (Guckenheimer and Holmes (1983); Kuznetsov (1998)). However, because of the lack of differentiability in piecewise smooth systems the above mentioned dimension reduction technique does not hold. Consequently, our knowledge about the higher dimensional piecewise smooth systems is far from satisfactory.

In smooth maps, a bifurcation occurs when the eigenvalues (real or complex conjugate) of a fixed point (or a periodic orbit) cross the unit circle smoothly. However, in nonsmooth maps, a border collision bifurcation takes place when a fixed point (or a periodic orbit) crosses or collides with the border resulting in an abrupt change of the eigenvalues (real or complex conjugate) from inside of the unit circle to the outside of it. Although, piecewise smooth maps appear simple, they may exhibit very complicated dynamics (Banerjee and Verghese (2001); Zhusubaliyev and Mosekilde (2003); di Bernardo et al. (2007); Simpson (2010)).

The local dynamical behavior of piecewise smooth maps is often described by linear terms of an appropriate Taylor series expansion. This yields two immediate advantages. First, the piecewise linear approximation of the smooth, nonlinear system is relatively easier to study. Second, the size of the invariant sets created at a bifurcation grows linearly with respect to the parameter. In the literature, most of the studies on nonsmooth discrete dynamical systems have been conducted using the normal form maps. A normal

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form nonsmooth map is an appropriate local linear approximation describing the systems dynamics in the neighborhood of the grazing point (di Bernardo et al. (2001a,b)). A grazing takes place when a periodic orbit of a piecewise smooth flow hits one of its discontinuity boundaries tangentially. Feigin (1970) conjectured that the dynamics near grazing of a general n-dimensional piecewise smooth system can be described by piecewise smooth linear local maps. Thus, under an appropriate choice of coordinates, the grazing of a limit cycle of the original piecewise smooth system corresponds to the border collision of its normal form. This idea was independently used by Nusse and Yorke (1992) to derive the normal form. The detailed derivation of the normal form in one-dimension is available in Banerjee et al. (2000a) and that in two-dimension is available in Yuan (1997) and Banerjee and Grebogi (1999).

Nevertheless, the normal form maps are not always linear for many practical dynamical systems. For example, an impact oscillator is governed by so called two-dimensional *Nordmark map* (Nordmark (1991); Fredriksson and Nordmark (1997)), which contains a *square-root singularity* on one side of the border and linear behavior on the other. To analyze stick-slip oscillation, Dankowicz and Nordmark (2000) defined a normal form map with a 3/2-order singularity. These types of maps are locally differentiable at the bifurcation point but have additional nonlinear terms acting on one side of the border so that higher derivatives of these maps are not smooth. These maps arise for grazing or sliding bifurcations (di Bernardo et al. (2001a,b, 2008)).

A lot of theoretical effort had been invested to study the local border collision bifurcations in piecewise smooth linear maps which are continuous across the border. Feigin (1970) derived three elementary conditions for the existence of period-1 and period-2 solutions before and after the border collisions for general n-dimensional piecewise smooth maps. The three elementary conditions were given in terms of the product of the characteristic polynomials of the map dynamical matrices and the same set of conditions

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was also presented in terms of the map eigenvalues on each side of the border. However, Feigin did not address the stability of the orbits, and the occurrence of high perodic orbits and chaos. In order to obtain a complete classification, one has to consider the stability of the orbit together with their existence. The classification of border collision bifurcations is complete only for one-dimensional piecewise smooth normal form map (Banerjee et al. (2000a)). Concerning two-dimensional piecewise smooth linear normal form map, Nusse and Yorke (1992, 1995) gave a general classification for the occurrence of border collision bifurcations based on the concept of the *orbit index* (Mallet-Paret and Yorke (1982)). Apart from that, they had derived the normal form for the two-dimensional piecewise smooth map, which was later adopted in the development of the theory of border collision bifurcations. Later, Banerjee and Grebogi (1999) proposed a classification of border collision bifurcations assuming that the map is dissipative, i.e., in both sides of the border the determinant of the Jacobian matrix is less than unity. They had also demonstrated the application of the theory, both in one- and two-dimensions, in explaining the dynamics of power electronics switching circuits (Banerjee et al. (2000a,b)).

Gradually the efforts were increasingly directed to study the two-dimensional piecewise smooth map. It had been found by Banerjee et al. (1998) that robust chaos can occur in two-dimensional piecewise smooth maps. A chaotic attractor is robust if, for its parameter values, there exists a neighborhood in the parameter space where there is no periodic windows. Kowalczyk (2005) presented a proof of the onset of attractors which are robust to small parameter changes for a noninvertible map derived by Parui and Banerjee (2002). Recently, Elhadj and Sprott (2008) gave an overview on some issues of common concern related to the robustness of chaos both in smooth and non-smooth dynamical systems with applications to real world systems. Nonsmooth maps can exhibit a strange type of bifurcation in which multiple coexisting attractors are

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created or destroyed simultaneously (Kapitaniak and Maistrenko (1998); Dutta et al. (1999); Zhusubaliyev et al. (2008a); Avrutin et al. (2009b)). The main feature of these bifurcations is that they lead to unpredictable behavior of trajectories when a system parameter is slowly varied through the bifurcation point. The exotic thing that can happen in two-dimensional piecewise smooth map is the so called *dangerous bifurcation* related to the case in which a fixed point is stable before and after the border collision bifurcation while at the border the dynamics is divergent (Hassouneh et al. (2004)). However, this does not take place in one-dimensional piecewise smooth maps. Ganguli and Banerjee (2005) reported the analytical conditions depending on the parameters for which dangerous border collision bifurcation may occur. Dangerous border collision bifurcation also have been described for simplified two-dimensional nonsmooth map in Do and Beak (2006), Do (2007) and Do et al. (2008). Recently, in an impact oscillator (Ma et al. (2008); Ing et al. (2008)), this bifurcation has been observed by Banerjee et al. (2009). In the two-dimensional normal form map, the existence of snap-back repeller has been first observed by Glendinning and Wong (2009) and later by Gardini and Tramontana (2010). A snap-back repeller is a type of homoclinic orbit which can only exist in non-invertible maps, and which intimate the existence of unstable chaotic motion (Marotto (1978, 2005)). The discovery of the border collision bifurcation *occurring at infinity* (so called Poincaré equator collision) in a piecewise smooth map (Avrutin et al. (2010c)) is also an important development in piecewise smooth systems.

Apart form the studies on normal form two-dimensional piecewise linear maps, several researchers have studied nonsmooth bifurcations in other forms of piecewise linear, continuous maps. Parui and Banerjee (2002) developed the theory of border collision bifurcation for the special case where the state space is piecewise smooth, but two-dimensional in one side of the borderline, and one-dimensional in the other side. Multistability, arithmetically period-adding bifurcations and existence of homoclinic orbits embedded in a

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non-hyperbolic invariant set in the piecewise linear map were also investigated (Beak and Do (2009); Do and Lai (2008)). Response of a subharmonic perturbation to the bifurcation parameter at a border collision period-doubling point for general piecewise linear continuous map was investigated by Zhao and Schaeffer (2007). The resulting behavior was characterized quantitatively by a gain, which is the ratio of the response amplitude to the applied perturbation amplitude. The gain in a border collision period-doubling bifurcation was found to be qualitatively different in respect of dependence on parameters from that of a smooth period-doubling bifurcation. Simpson and Meiss (2009b) have unfolded the co-dimension-2 simultaneous occurrence of a border collision bifurcation and a period-doubling bifurcation for a general piecewise smooth, continuous map. In a *stochastic nonsmooth map* (map with additional noise term) Griffin and Hogan (2005) explored the blurring of bifurcation boundaries, apparent loss of bistability, stochastic resonance and an advance or delay of bifurcations. For higher dimensional nonsmooth maps, current studies are limited to very general results on the existence of *change of solution* types border collision bifurcation (di Bernardo et al. (1999); Roy and Roy (2008)).

The previous section considered piecewise linear continuous maps. However, as mentioned above there exist piecewise smooth maps which can have both linear and nonlinear terms. Most of the normal form maps of grazing and sliding bifurcations in planner piecewise smooth systems have nonlinearities (e.g., square-root singularity, 3/2-singularity), they are less studied in comparison with the piecewise smooth linear maps. The key aspect of square-root singular maps is that the square-root term leads to infinite local stretching on the one side of the border. A one-dimensional map of this type was studied by Nusse et al. (1994), Foale and Bishop (1994) and Halse et al. (2003). In recent past, Avrutin et al. (2010a) have studied a similar map which has both linear as well as square-root term in one side of the border. They have reported the occur-

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rence of *basin boundary metamorphoses* and detected the organizing centers of several typical nonsmooth bifurcation sequences. Botella-Soler et al. (2009) have analyzed a one-dimensional piecewise smooth map proposed originally in studies on population dynamics. Their map is composed of a linear part and a power-law decreasing part, and has three parameters. Border collision bifurcations in two-dimensional piecewise smooth maps with square-root singularity were reported in Chin et al. (1994, 1995), Casas et al. (1996) and Molenaar et al. (2001). The bifurcation scenarios which were investigated in these maps are as follows: existence of period-increment cascades, chaotic bands sandwiched between two periodic windows, period-adding cascades, etc. The bifurcation studies on piecewise smooth maps with 3/2-singularity and higher order nonlinearity are in a very preliminary stage and only some results are available, which were reported in Halse et al. (2003), Sushko et al. (2005, 2006) and Dutta and Bhattacharjee (2008).

There is now an enormous volume of research work on the piecewise smooth continuous maps (Banerjee and Verghese (2001); Zhusubaliyev and Mosekilde (2003); di Bernardo et al. (2007, 2008); Simpson (2010)). However, discontinuous maps (or maps with gaps) have received less attention even though many switching dynamical systems, for example, the Colpitts oscillator (Maggio et al. (2000)), dc-dc converters (Banerjee et al. (2004); Kapat et al. (2010)), thyristor controlled reactor circuits (Rajaraman et al. (1996)), sigma-delta modulators (Feely and Chua (1992)), digitally controlled systems (Haller and Stepan (1996)), and many other electronic circuits (Sharkovsky and Chua (1993)) give rise to discontinuous maps. The earliest work in this direction was done by Keener (1980). He considered an interval map which shows the chaotic behavior. It has been shown that such discontinuous maps exhibit period-adding bifurcations (Feely and Chua (1992)), type-V intermittency (Bauer et al. (1992); He et al. (1992)), multiple devil's staircase (Qu et al. (1998)), etc. Lofaro (1996) had observed period-increment cascade in these maps. Later, Jain and Banerjee (2003) presented the first attempt to classify the

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border collision bifurcations that occur in one-dimensional piecewise-linear discontinuous maps. They also illustrated the application of the theory in a switching circuits (Jain and Banerjee (2003); Banerjee et al. (2004)). Further works on the dynamics of the discontinuous map proposed by Jain and Banerjee (2003) were investigated in Avrutin and Schanz (2006, 2008) and Avrutin et al. (2006, 2007b,a, 2008a,b, 2009a). They specifically studied the multiparametric bifurcation phenomena and reported so-called *big bang bifurcation, band-count adding*, etc. The main result regarding the existence of the period-1 fixed point in general n-dimensional discontinuous maps was reported in Hogan et al. (2007), and subsequently in Routroy et al. (2006) and Dutta et al. (2008). Moreover, Dutta et al. (2008) derived the existence condition for period-2 orbit and as an application example examined the two-dimensional discontinuous map. Recently, Bischi et al. (2010) have studied a one-dimensional map composed of three linear pieces which are separated by two discontinuity points, motivated by a model arising in social sciences. The dynamics of quadratic maps with a gap defined on the unit interval was also reported (Chia and Tan (1991, 1992); Tan and Chia (1993); Avrutin and Schanz (2004, 2005)). Pring and Budd (2010) studied discontinuous maps which arise in cam-follower system (Osorio et al. (2008)) and non-smoothly forced impact oscillators (Budd and Piiroinen (2006)).

The Russian mathematician Leonov (1959, 1960a,b) had presented a number of fundamental results on piecewise-linear discontinuous maps. He considered a four-parametric family of one-dimensional piecewise linear maps with a single discontinuity. It was demonstrated that the number of parameters which determine the dynamical behavior can be reduced to three and the bifurcation structure of that map in the complete three-dimensional parameter space was reported. The bifurcation scenarios now known as period-adding and period-incrementing was also observed and identified. Not only the existence boundaries of specific periodic orbits were explained by introducing the bor-

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der collision bifurcations, but also the bifurcation scenarios formed by these orbits were reported. To simplify the calculation of border collision bifurcation curves of higher periodic orbits in piecewise linear systems, he developed a basic idea called *Leonov's approach*. Avrutin et al. (2010b), Avrutin et al. (2010d) and Gardini et al. (2010) rediscovered Leonov's approach and they named it as *map replacement approach*. They demonstrated the basic idea and explained what are the pre-condition for its application and self-similarity of the period-adding structure by taking one-dimensional maps as examples. The map replacement approach for the calculation of crisis bifurcation curves and band count adding structure was developed by Avrutin et al. (2010d). It was necessary to extend these results for other bifurcation scenarios and for general piecewise smooth maps of arbitrary dimensions. A part of this thesis is devoted to extend the map replacement approach for general two-dimensional piecewise linear maps.

As shown by Nusse et al. (1994), Avrutin and Schanz (2000) and Dutta and Banerjee (2009), the value of the scaling constant δ of the period-increment cascade of piecewise smooth maps depends on the parameter of the system. Hence, by choosing the parameter appropriately, it is possible to adjust the value of the scaling constant. This has a direct consequence of the functional renormalization group equation of the period-increment scenario. Recently, Glendinning (2008) has presented some important results in this direction by considering two classes of one-dimensional maps with single discontinuity.

Together with the period-doubling and intermittency transitions, torus destruction represents one of the classical routes to chaos in dissipative systems. It has attracted considerable interest because it may happen in various physical systems, for example, coupled oscillator systems in physics (Mosekilde et al. (1990); Bakri et al. (2004)), biology (Hayashi et al. (1982); Baier et al. (1993)) and other fields of science (Matsumoto et al. (1987); Anishchenko et al. (1993, 1994); Pereira et al. (2009)). In smooth systems, the appearance of chaos following the breakdown of two-dimensional torus has been in-

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vestigated theoretically by many researchers (*some notable references are* Landau (1944); Hopf (1948); Ruelle and Takens (1971); Curry and Yorke (1977); Newhouse et al. (1978)). Ruelle and Takens (1971) specified a finite bifurcation sequence in the quasiperiodicity route to chaos (2D torus → 3D torus → chaos). For a specific mathematical model using numerical analysis, Curry and Yorke (1977) had described another route in which chaos appears directly from the destruction of a 2D torus (without appearance of third frequency). Later, Afraimovich and Shilnikov (1991) have found three basic routes to chaos. They outlined three fundamental routes to chaos via torus breakdown for smooth dynamical systems. These are, (i) period-doubling bifurcations of the phase-locked limit cycles, (ii) saddle-node bifurcation in the presence of a homoclinic structure, and (iii) soft transition due to the loss of torus smoothness. The possible topological transitions of torus for loss of torus smoothness have been overviewed by Aronson et al. (1982), Anishchenko et al. (1993), Maistrenko et al. (2003) and Kuznetsov (1998). The torus destruction routes to chaos have been confirmed for many practical systems, for example, Chua's torus circuit (Anishchenko et al. (1993), driven double scroll circuit (Baptista and Caldas (1998)), dc-dc converter (Aroudi et al. (1999, 2000); Aroudi and Leyva (2001)), glow discharge plasma (Letellier et al. (2001)) and van der Pol system (Letellier et al. (2008)).

Two-dimensional piecewise smooth maps have been mainly considered in dissipative cases associated with the real eigenvalues of the fixed point which undergoes the border collision bifurcation (Banerjee and Grebogi (1999); di Bernardo et al. (1999)). However, in recent past it has been found that a large number of practical systems yield piecewise smooth maps with complex conjugate eigenvalues (Zhusubaliyev and Mosekilde (2003); Zhusubaliyev et al. (2003); Gallegati et al. (2003); Sushko and Gardini (2006); Zhusubaliyev and Mosekilde (2006a); Zhusubaliyev et al. (2006)). In a series of recent publications, Zhusubaliyev and Mosekilde (2006a,b,c, 2008b,a), Zhusubaliyev et al.

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(2006) and Maity et al. (2007) have shown that at a bifurcation point the complex conjugate multipliers associated with the fixed point jump from the inside of the unit circle to the outside of it. The resulting bifurcation is a nonsmooth analogue of the classical Neimark-Sacker bifurcation. At this bifurcation point an invariant circle may be created which is associated with the quasiperiodic or mode-locked dynamics. Mode-locking on the invariant circle corresponds to a point in a resonance tongue (Yang and Hao (1987)). As a parameter of the map is varied, the invariant circle may be destroyed. The birth of quasiperiodicity through border collision bifurcation has been developed by Zhusubaliyev et al. (2006). Using a dc-dc converter with two-level control as an example, they reported the first experimental verification of the direct transition to quasiperiodicity through a border collision bifurcation.

The closed invariant curve forming the mode-locked torus is defined by the closure of unstable manifolds of a saddle cycle and the points of the saddle cycle and stable cycle of same periodicity. Homoclinic (or heteroclinic) tangles arise when the unstable manifold of a saddle intersects the stable manifold of that (or another) saddle cycle (Kuznetsov (1998)). The appearance of homoclinic tangles implies the existence of Smale horseshoe (Smale (1967, 1998)) dynamics as a consequent occurrence of chaos. The torus destruction through homoclinic (or heteroclinic) bifurcation is also possible in two-dimensional piecewise smooth maps. Global analysis of the closed invariant curve for two-dimensional smooth maps have been reported by Agliari et al. (2006). However, for the two-dimensional piecewise smooth normal form map, the torus breakdown routes to chaos remain to be investigated. Transitions to chaos via torus breakdown in the two-dimensional piecewise smooth maps are different from those in the smooth maps because the modification arises in connection with the border collision bifurcation.

Recently, Sushko and Gardini (2008) have illustrated center bifurcation in a two-dimensional piecewise smooth map, which occurs when a fixed point with complex

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conjugate eigenvalues passes through the unit circle and the determinant of the associated Jacobian matrix is unity. Further research on piecewise smooth maps with complex eigenvalues was conducted by Simpson and Meiss (2008a, 2009a, 2010). Specifically they have explored the codimension-2 shrinking point bifurcations. In contrast to smooth systems, piecewise smooth maps typically exhibit a distinctive lens-chain (or sausage) structure. The connecting point of any two adjacent lenses in a lens-chain structure is called shrinking point where the width of the resonance tongue is zero.

Many areas and applications give rise to systems that can be described by maps which are noninvertible, for example, adaptive control system (Adomaitis and Kevrekidis (1991); Frouzakis et al. (1991)), economics (Agliari et al. (2003)), neural networks (Rico-Martinez et al. (2000)), etc. The properties of the two-dimensional noninvertible maps, such as the interactions between invarient set and critical curve have been developed by Mira et al. (1996) and Frouzakis et al. (1997). Basin bifurcations and fractalization of basins of the non-invertible maps have also been focused in many papers (Mira et al. (1994); Agliari et al. (2003); Bischi et al. (2006)). Gardini (1996) addressed transverse homoclinic orbits for planer maps. England et al. (2005) illustrated the bifurcation theory of stable sets in two-dimensional noninvertible planer maps. The existence of homoclinic tangles and the associated dynamic behavior for noninvertible maps have been developed by Sander (2000). Lerman and Shilnikov (1988) have established the conditions for a transverse homoclinic point of such maps to be contained in a horseshoe. The appearance of chaos through torus breakdown in two-dimensional noninvertible maps has been proposed by Lorenz (1989) and Maistrenko et al. (2003). Later, Frouzakis et al. (2003) re-investigated torus breakdown route to chaos proposed by Lorenz (1989). Recently, Zhusubaliyev and Mosekilde (2008c, 2009b,a) described different scenarios of formation and destruction of multilayered tori in a system of two coupled non-invertible maps. Much of the above papers have encountered the smooth non-invertible maps.

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However, the torus breakdown routes to chaos in the case of maps where the system function is piecewise-linear as well as nonivertible remains to be investigated.

In most of the studies on the effects of nonlinearity and nonsmoothness of switching systems, the essential method has been to sample the states at discrete intervals of time and to analyze the observed bifurcation scenarios in terms of the theory of border collision bifurcations in discrete time systems or maps. Apart from map-based modelling, bifurcations in piecewise smooth continuous time dynamical systems or flows were also widely studied. Such bifurcations were sometimes called "discontinuous bifurcations" (Leine et al. (2000); Leine and van Campen (2002, 2006); Leine and Nijmeijer (2004); Colombo and Dercole (2010)). It was shown that piecewise smooth continuous time systems exhibit a variety of possible co-dimension-1 and co-dimension-2 border collision bifurcations as equilibrium points accross the hyper-surface separating the regions of smooth behavior as a system parameter is slowly varied through a critical value (for a recent review see di Bernardo et al. (2008)). A strategy for the classification of co-dimension-2 grazing bifurcations of limit cycles in piecewise smooth systems of ordinary differential equations was proposed in Kowalczyk et al. (2006). Simpson and Meiss (2007, 2008b) reported bifurcations in nonsmooth systems that are reminiscent of the classical Andronov-Hopf bifurcations in smooth systems. The simultaneous occurrence of a discontinuous bifurcation and a smooth Andronov-Hopf bifurcation has recently been observed in a model of yeast growth (Simpson et al. (2009)).

1.4 Objective and Scope of This Thesis

The main objective of this thesis is to study the local and global bifurcation phenomena of various forms of piecewise smooth maps (Banerjee and Verghese (2001); Zhusubaliyev and Mosekilde (2003); di Bernardo et al. (2007); Simpson (2010)). For a general piece-

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wise smooth map, in most of the cases it is possible to linearize the map functions about the bifurcation point, and in many cases it is also possible to define a co-ordinate transformation that transforms the map into its normal form. Most of the studies on nonsmooth dynamical systems were carried out by considering the normal forms, which played an important role in the development of nonsmooth bifurcation theory. The present study is motivated by that approach.

The derivation of bifurcation curves is invaluable in order to understand the complex structure of multiparameter bifurcation diagrams. In the study of nonsmooth maps, such calculation tends to become cumbersome. Recently (Avrutin et al. (2010b,d)) tackled this problem in a one-dimensional piecewise-linear discontinuous map by generalizing the map replacement technique developed by Leonov (1959, 1960a,b). The natural question was: Can the technique be extended to a two-dimensional piecewise-linear map to evaluate the border collision bifurcation curves and stability boundaries analytically? This constitutes the first problem we have sought to take up in this thesis.

Recent work by Zhusubaliyev et al. (2006) has shown that torus formation in piecewise smooth maps can take place through a nonsmooth Neimark-Sacker bifurcation in which a pair of complex conjugate eigenvalues of a stable cycle abruptly jump from the inside of the unit circle to the outside of it. They have determined the transition from a mode-locked torus to a quasiperiodic torus. Numerical investigation shows that further parameter variation leads to the transition to chaos, but the mechanism of that transition was not known. In this thesis, we investigate the transition to chaos through the destruction of a mode-locked torus in an invertible, two-dimensional piecewise smooth map. This is the second problem we take up in this thesis.

It has been shown that the routes to chaos through mode-locked torus breakdown in noninvertible smooth maps are different from those in the invertible smooth maps

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due to an important role played by the noninvertibility (Maistrenko et al. (2003)). For noninvertible smooth maps, two new characteristic shapes of torus were reported: one is *cusp torus* and another is *loop torus*. In the context of nonsmooth systems it was not known if the same mechanisms are effective in causing the breakdown of the torus. In this thesis, we have taken a first step to answer this question.

Once the bifurcation theory of one-dimensional and two-dimensional maps are developed to a satisfactory level, the next natural step is to investigate the dynamical phenomena of higher dimensional piecewise smooth maps (di Bernardo et al. (2007)). While there are practical dynamical systems which can be modeled by three-dimensional piecewise smooth maps (Zhusubaliyev and Mosekilde (2006a); Zhusubaliyev et al. (2007); Zhusubaliyev and Mosekilde (2003)), very little investigation has been carried out on such systems. Only the normal form has been derived and the stability criteria of period-1 fixed points are available in literature (Roy and Roy (2008)). Development of the local and global bifurcation theory for such a map is the fourth problem we investigate in this thesis.

1.5 Outline of the Thesis

An outline of the rest of the thesis is as follows.

In ***Chapter 2***, we have demonstrated two important steps which show the power of the map replacement approach. First, we establish that the applicability of this approach is not restricted to one-dimensional maps. Here, we consider the general two-dimensional piecewise-linear map with discontinuity at the border and as an additional application example the two-dimensional normal form piecewise smooth map.

In the second generalization step, we have reported that in addition to the well known border collision bifurcations, the map replacement approach is also applicable for the

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calculation of border collision bifurcations occurring at infinity and degenerate period-doubling bifurcations. In particular, for the general map using the map replacement approach we have calculated the bifurcation curves for the families of periodic orbits with first and second levels of complexity.

In ***Chapter 3***, we have analyzed three mechanisms of transitions to chaos via resonance torus destruction in a two-dimensional invertible piecewise smooth map. In one of the routes reported in this chapter, a homoclinic intersection first destroys the torus. In the absence of the torus, the stable node undergoes a period-doubling, immediately followed by a border collision bifurcation that gives birth to the chaotic orbit. In another route, the first homoclinic tangency is followed by a second homoclinic tangency, which gives birth to a single-band chaotic attractor. But the stable periodic orbit persists. At a different parameter value, this periodic orbit undergoes a period-doubling bifurcation, again immediately followed by a border collision bifurcation resulting in a different multiband chaotic attractor. The multiband attractor is destroyed at a nonsmooth saddle-node bifurcation, where there is a hard transition from one chaotic orbit to another. In the third route, the first homoclinic tangency is followed by a second homoclinic tangency and a chaotic attractor is born. At a specific parameter value, the stable node or focus collides with the saddle on the border and both are destroyed through a nonsmooth fold bifurcation. All the three scenarios are demonstrated in detail with the help of numerically simulated one- and two-parameter bifurcation diagrams, and phase portraits.

In ***Chapter 4***, we have studied a two-dimensional noninvertible piecewise smooth map and have shown that in this map the destruction of a closed invariant curve defining the mode-locked torus leads to new characteristic phenomena as compared to that of the invertible nonsmooth as well as smooth maps. Owing to the noninvertibility in the map, a point can have two distinct preimages and the unstable manifolds may have stable self-

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intersections, resulting loops on the invariant curve. Therefore, at some parameter value, the mode-locked torus becomes a loop torus (a torus formed by a closed invariant curve consisting of loops on it). As our map is linear in both sides of the border and the critical curve is the x-axis which has no cusp point, so cusps do not appear on the invariant curve unlike smooth noninvertible maps. The invariant curve is destroyed through a homoclinic bifurcation. After the first homoclinic tangency, the stable manifolds become closed curves when one of the points of these lies on the critical curve. Immediately after the second homoclinic tangency, the basin of attraction of the attracting cycle changes from connected to nonconnected basin.

Chapter 5 presents local and global bifurcation analysis in a three-dimensional piecewise smooth map. Regarding the local bifurcations, we have made a detailed investigation on the nature of eigenvalues and calculated the saddle-node, period-doubling and Neimark-Sacker bifurcation surfaces. Based on the Feigin's classification strategy, the conditions for the occurrence of local border collision bifurcations are also derived. It is reported that the dangerous border collision bifurcation can also occur in this map.

The transitions from a mode-locked torus to a quasiperiodic torus or chaos are determined, varying the parameters from the inside of a resonance tongue to the outside of it. Since the map is three-dimensional we have studied two cases, first when the map is dissipative in one side of the border and expanding in the other side, and second when the map is dissipative in both sides. For the first case, we have studied the breakdown of a mode-locked torus via a degenerate period-doubling bifurcation and homoclinic bifurcation. For the second case, we have investigated a mode-locked torus breakdown via homoclinic bifurcation and transition from a mode-locked torus to a quasiperiodic torus without any global bifurcation.

Chapter 6 summarizes the contributions of this thesis and indicates the further

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