Chapter 1

Introduction and Summary

1.1 Introduction

Estimation of reliability of an equipment is of paramount importance in the context of modern technology and its future development. When we buy an equipment we expect it to function properly for a reasonable period of time. When a new brand of a product is floated in the market, the customers will like to have some information about the average life of the products. Since the item is likely to fail at any time, it is quite customary to assume that life of the item is a random variable with a distribution function $F(t)$, that is, the probability that the item fails before time t. This type of questions, can be answered if we know $F(t)$. Let $R(t) = 1 - F(t)$ denote the probability of failure free operation until time t. However the parameters of these life time distributions are unknown and so the behavior of the component can be investigated by estimating the reliability of the component. The components may be mechanical, electronic or of any other type. Measurements include failure rates and system reliability. In many cases, failure of an equipment is inconvenient but not dangerous, such as failure of a television, airconditioner, fan or computer etc. In other cases it can be life threatening, for example, failure of an equipment during operation may cause death of a patient. Failure of electronic control of an aircraft may cause the plane to crash. We may be interested in different measures of reliability when we deal with various components. For example, the measure associated with components of a nuclear power reactor is frequently the hazard rate or the failure rate as instantaneous failure rate of a nuclear power reactor is of primary concern. On the other hand a power supply system for a deep space satellite must function for the entire mission period which requires the probability of survival.

The present thesis deals mainly with the estimation of reliability and hazard rate functions under different exponential life models. Some useful definitions in reliability studies are given below.

Definition 1.1 Let the random variable X represent the life time or time to failure of a component. The probability that a component survives until time $t(>0)$ is called the reliability of the component at time t and is defined by

$$R(t) = P(X > t) = 1 - F_X(t), \tag{1.1.1}$$

where $F_X(x)$ is the cumulative distribution function (cdf) of X.

We expect our cars, computers, electrical appliances, television etc. to function for a specified period of time without failing. One needs to know the reliability of these systems.

Definition 1.2 (Series system): Let $X_1, \ldots, X_k$ be independent and denote the life times of a k-component series system, then the system fails if at least one of the components fails. So, the system reliability is defined as

$$R(t) = P(\min X_i > t) = \prod_{i=1}^{k} P(X_i > t). \tag{1.1.2}$$

For example, suppose a CPU has two microprocessors connected in series. Let the lives of microprocessors be independent with reliabilities at time t as $R_1(t)$ and $R_2(t)$ respectively. Then the reliability of the CPU is $R_1(t)R_2(t)$.

Definition 1.3 (Parallel system): Let $X_1, \ldots, X_k$ be independent and denote the life times of a k-component parallel system, then the system fails if all the components fail. So, the system reliability at time t, is defined as

$$\begin{aligned} R(t) &= P(\max X_i > t) \\ &= 1 - P(\max X_i \leq t) \\ &= 1 - \prod_{i=1}^{k}(1 - P(X_i > t)). \end{aligned} \tag{1.1.3}$$

For example, if an office has k copy machines, it is possible to copy a document if at least one machine is in good condition.

Definition 1.4 Let X and Y be two continuous (discrete) random variables denoting the stress and strength respectively of a mechanical equipment, then the system survives if stress is less than the strength. In this case the reliability R of the system is defined by

$$\begin{aligned} R = P(X < Y) &= \int_{x<y}\int f(x,y)dxdy, \text{ when X and Y are continuous,} \\ &= \sum_{x<y}\sum f(x,y), \text{ when X and Y are discrete,} \end{aligned} \tag{1.1.4}$$

where $f(x,y)$ is the joint probability density (mass) function of X and Y.

There are some applications of stress-strength models for the k components system and which have a common stress. Suppose different electric bulbs are arranged in a series and there is a common voltage passing through them. If the common voltage is more then the strength of the filament of any bulb, then the system fails. Voltage of a current is a continuous variable. This type of stress-strength models are quite common in industrial applications.

Definition 1.5 The hazard rate or failure rate function is defined as the instantaneous failure rate of a component given by

$$H(t) = \lim_{\triangle t \to 0} \frac{P(t < X \leq t + \triangle t)}{\triangle t P(X > t)}.$$

If $R(t)$ and $f(t)$ denote the reliability function and density function of X, respectively, then

$$H(t) = \frac{f(t)}{R(t)} = \frac{-R'(t)}{R(t)} = -\frac{d}{dt}(\log R(t)). \tag{1.1.5}$$

1.2 A Review of Literature on Estimation of Measures of Reliability

In this section we describe the details of earlier results on various measures of reliability.

1.2.1 Estimation of Reliability Function

A significant amount of literature is available in the field of reliability estimation, especially when the life times are assumed to be exponentially distributed. One may refer to Sinha (1986) for detailed review of results in this area. Pugh (1963) obtained the uniformly minimum variance unbiased estimator (UMVUE) of reliability in the exponential case. Basu (1964) considered the estimation of reliability function when the life time X

follows a negative exponential distribution with mean θ. The reliability function is given by

$$R(t) = P(X > t) = e^{\frac{-t}{\theta}},\ \theta > 0,\ t > 0.$$

Individual components are subjected to life testing and the test continues until a preassigned number r_0 of failures have occurred. Basu derived the uniformly minimum variance unbiased estimator (UMVUE) of the reliability function as

$$\hat{R}_U = \left(1 - \frac{t}{T}\right)^{r_0 - 1} I_{(t,\infty)}(T), \tag{1.2.6}$$

where T is the total life on all the components tested and $I_B(z)$ is the usual indicator function. See also Voinov and Nikulin (1993).

Rutemiller (1966) considered a multi-component system with k identical components. The lifetimes of these components are exponentially distributed. He considered the problem of finding an unbiased estimator for the system reliability of k components arranged in series and parallel, respectively. Suppose individual components have been placed on test and the test continued until a preassigned total life T_0 ($T_0 > t$) has been observed. Let the number of failures in time T_0 be r. Then the UMVUEs of the system reliability for both parallel and series systems are given by

$$\text{(Parallel)}\ \ \hat{R}_{U1} \ = \ \sum_{j=1}^{k} (-1)^{j-1}\, {}^{k}C_j\ (1 - \frac{jt}{T_0})^r,\ \ I_{(kt,\infty)}(T_0) \tag{1.2.7}$$

and

$$\text{(Series)}\ \ \hat{R}_{U2} \ = \ (1 - \frac{kt}{T_0})^r,\ \ I_{(kt,\infty)}(T_0), \tag{1.2.8}$$

respectively. Although the maximum likelihood estimator (MLE) and UMVUE of reliability are easy to compute when the life times are exponentially distributed, the expressions for the mean squared errors (MSE) are complicated.

Bhattacharya (1967) introduced a Bayesian approach to life testing and reliability. Some proper and improper priors are proposed and the corresponding Bayes and generalized Bayes estimators were obtained.

Zacks and Even (1966a) gave the exact mathematical expressions for MSE of both the estimators for small sample sizes of one unit systems. They also considered the efficiencies in small samples of these two estimators. The calculations are done for exponential, normal and Poissions distributions. Numerical comparisons of the mean squared errors are carried out.

Zacks and Even (1966b) have also derived the UMVUE and the MLE of reliability functions for a two components system connected in series as well as in parallel for exponential and Possion models. The UMVUE $\hat{R}_{U3}$ for the exponential life time model at time t (> 0) was obtained as

$$\begin{aligned} \hat{R}_{U3} &= \left(1 - \frac{2t}{T}\right)^{2n-1}, \; I_{(\frac{t}{2},\infty)}(T), \quad \text{when } \theta_1 \text{ and } \theta_2 \text{ are assumed to be equal,} \\ &= \prod_{i=1}^{k}\left(1 - \frac{t}{T_i}\right)^{n_i-1}, \; I_{(t,\infty)} \min(T_1, T_2), \\ &\quad \text{when } \theta_1 \text{ and } \theta_2 \text{ are assumed to be unequal,} \end{aligned} \tag{1.2.9}$$

where θ_1 and θ_2 denote the expected lives of the two components.

Basu and Mawaziny (1978) have extended this work to a k-out-of-m system in the independent exponential case. They obtained the UMVUE for both distinct and identical parameter cases. The performance of the estimator when the component life times are identically distributed is compared with the corresponding MLE for both large and small sample sizes using Monte-Carlo simulation. They showed that these estimators are asymptotically equivalent. They also proved that the UMVUE has asymptotically normal distribution.

Chao (1981) derived approximate expressions for mean squared errors of the estimators obtained by Basu and Mawaziny (1978). Kurkijian et al. (1987) considered the estimation reliability function of one parameter exponential distribution. They propose two new estimators and their risk performances are numerically compared with MLE and UMVUE. It is shown that the new estimators have better risk performances over a portion of the parameter space.

Singh et al. (2004) have considered a convex combination of MLE $\hat{R}_M(t)$ and UMVUE $\hat{R}_U(t)$ of the reliability function of an exponential distribution.

$$R_a(t) = a\hat{R}_U(t) + (1-a)\hat{R}_M(t), \; -\infty < a < \infty. \tag{1.2.10}$$

They derived the choice of a that minimizes the MSE of $R_a(t)$. As this choice depends on the parameter, they replace the parameter by the MLE. The resulting estimator is compared numerically with the MLE and the UMVUE.

Gupta and Gupta (1987) estimated the reliability function in the following models.

(i) $N(\mu, 1)$ (Normal model) , with σ known and equal to 1.

(ii) $N(\mu, \sigma^2)$ (Normal model), with σ is unknown.

(iii) $E(X, \lambda, \theta)$ (Exponential Model), with unknown location and scale parameters.

Five estimators were proposed. These estimators are (i) MLE, (ii) UMVUE, (iii) an estimator obtained through structural approach, (iv) Bayes estimator, (v) an estimator from predictive approach. The mean squared errors of these estimators are compared analytically whenever possible. In other cases simulation studies are carried out to compare these five estimators.

Dey et al. (1988) considered a series system with p components which have exponential life times. They considered the problem of estimating the system reliability using type II censored samples in exponential life time model. The system reliability is a function of $U(\underline{\lambda}) = \sum_{j=1}^{p} \lambda_j$, where λ_j is the hazard rate of the j^{th} component. An estimator of $U(\underline{\lambda})$ which dominates the MLE in terms of risk was derived. Dey and Jaisingh (1988) estimated the reliability function of a series system with independent components based on random samples from Weibull distribution with parameters θ_i and β_i. It was assumed that β_i's are known, which reduces the reliability of the system to a function of $\gamma(\underline{\theta})$, where $\underline{\theta} = (\theta_1, \ldots, \theta_p)$. An estimate of $\gamma(\underline{\theta})$ which is better than the UMVUE in terms of MSE was determined. The predicted reliability and the percentage of improvement for this estimator is compared and computed with the UMVUE of $\gamma(\underline{\theta})$.

Bhattacharya and Soejoeti (1989) developed a statistical model for step stress accelerated life test. It is motivated from the point of view that a change of stress has a multiplicative effect in the failure rate function over the remaining life. Properties of the proposed model, including an interpretation in terms of the conditional reliability and relationships with the existing models were discussed. MLEs of the parameters of this model were derived.

Basu and Ebrahimi (1992) derived the Bayes estimator of the reliability function in the exponential life testing model using an asymmetric loss function of the form

$$L(\triangle) = b[e^{a\triangle} - a\triangle - 1], \tag{1.2.11}$$

where $\triangle = (\frac{\hat{\theta}}{\theta} - 1)$, $a \neq 0$, $b > 0$. They have obtained the Bayes estimator of reliability function using various priors. A comparison of mean squared errors was made between these and with the Bayes estimates with respect to the squared error loss function.

Estimation of reliability, where X follows Pareto distribution was considered by Bhattacharya et al. (1994). The estimation based on the first r ordered observations and $n-r$ survivors was proposed. The UMVUE, Bayes estimators of the mean life, hazard rate and reliability function were derived.

The Bayes estimation of reliability function for a mixture of inverse Gussian distributions (IGD) has been studied by Akman and Huwang (1997) through a numerical approach. Since the calculations in deriving the Bayes estimators are difficult, they have used a rejection method to obtain them for all three parameters in this case. The model under their consideration is given by

$$f_p(x) = (1-p)f(x) + p\ g(x),\ p = 0, 1, 2, \tag{1.2.12}$$

where $f(y)$ is the density function of an inverse Gussian distribution with finite mean μ and $g(y) = y\frac{f(y)}{\mu}$.

Since the structures of the Bayes and natural estimators of reliability function are complicated, exact decision theoretic properties are hard to derive. Chaudhury et al. (1998) have studied the asymptotic decision theoretic properties of the natural estimators of reliability of a k components series system. The life times of these components are exponentially distributed. They established that $\hat{\theta}_{MLE}$ is second order admissible within a class of estimators of the form

$$\hat{\theta}_c = \hat{\theta}_{MLE} + \frac{1}{n}c(\hat{\theta}_{MLE}), \tag{1.2.13}$$

where $c(.)$ is assumed to admit Taylor's series expansion at θ.

El-Neweihi and Sinha (2000) provided a class of n unbiased estimators of the reliability function of an exponential distribution based on the first r observations of a ranked set sample of size n. The sample $X_{(11)}, \ldots, X_{(nn)}$ of independent but not identically distributed of ordered statistics is said to be ranked set sample. They conjectured that the estimator that uses all n observations in the sampling scheme has the minimum variance in the given class of n unbiased estimators. They proved this for $n = 2$ and 3. Ghitany (2005) has presented two counter examples to show that the conjecture made by El-Newihi and Sinha (2000) was incorrect. Ghitany (2005) has disproved the conjecture given by El-Neweihi and Sinha (2000) by taking $n = 4$ and 5, respectively.

Chaturvedi and Tomer (2003a) have considered the estimation of the reliability function and $P(X < Y)$ for the case of negative binomial distribution. The UMVUEs and the Bayes estimators of the reliability function and $P(X < Y)$ were derived. Using the UMVUEs and Bayes estimators of the powers of the parameter, the UMVUEs and the Bayes estimators were derived. The UMVU estimation of the reliability function and $P(X < Y)$ of the generalized life distributions were also studied by Chaturvedi and Tomer (2003b).

Nassar and Eissa (2004) have derived the Bayes estimates of the two shape parameters, reliability function and hazard rate for the exponentiated Weibull model (EWM) from complete and type II censored samples. The density of EWM is given by

$$f(x) = \alpha\theta x^{\alpha-1} e^{-x^{\alpha}} (1 - e^{-x^{\alpha}})^{\theta-1}, \quad x > 0,\ \theta > 0,\ \alpha > 0. \tag{1.2.14}$$

Conjugate priors for either one or two shape parameters are considered. An approximation form due to Lindley (1980) was used to obtain the Bayes estimates under squared error and linear-exponential (linex) loss function (1.2.11). The root mean squared errors of the estimates are computed. Comparisons are made between these estimates and MLE using a Monte-Carlo simulation study.

Bekker and Roux (2005) have derived the MLE, Bayes and empirical Bayes estimators of the truncated first moment and hazard rate function of the Maxwell distribution with density

$$f(x;\theta) = \frac{4}{\sqrt{\pi}}\ \theta^{\frac{3}{2}}\ x^2 e^{-\theta x^2}, \quad x \geq 0,\ \theta > 0. \tag{1.2.15}$$

A comparison of the relative efficiency of these three estimators is carried out using a simulation study.

1.2.2 Estimation of Hazard Rates and the Scale Parameters in Exponential Models

Sharma (1977) has considered the problem of estimating the hazard rate in a two parameter exponential distribution. The location and scale parameters are assumed to be unknown. Let $Y_1, \ldots, Y_n$ be a random sample from this population. Further let $\overline{Y}$ and Y_s denote the mean and the minimum of this sample, respectively, and $W = \overline{Y} - Y_s$. Based on a result of Kiefer (1957), Sharma claimed that the best affine equivariant estimator $\frac{n-3}{nW}$ is minimax. Further he obtained an improvement over the best affine equivariant estimator. The improved estimator is given by

$$\delta_1 = \begin{cases} \max\{\frac{n-2}{n(Y_s+W)}, \frac{n-3}{nW}\} & \text{if } Y_s > 0, \\ \frac{n-3}{nW}, & \text{otherwise.} \end{cases} \tag{1.2.16}$$

The problem of estimating the vector of scale parameters $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ and their reciprocals $(\theta_1, \ldots, \theta_p)$ of p gamma distributions was addressed by Berger (1980) using differential inequalities. Four types of losses of the form $\sum_{i=1}^{p} \theta_i^m (\delta_i\theta_i - 1)^2$, $m = -2, -1, 0$ and 1 were considered. Let $(X_1, \ldots, X_p)$ be independent and X_i follows a gamma distribution with the scale parameter θ_i^{-1} and shape parameter α_i, $i = 1, \ldots, p$. It was shown that $\underline{\delta}^*(X) = (\delta_1^*(X), \ldots, \delta_p^*(X))$, where

$$\delta_i^*(X) = \left(\frac{\alpha_i - 2}{X_i} + \frac{cX_i}{b + \sum_{j=1}^{p} X_j^2} \right), \ 0 < c < 4(p-1), \ b \geq 0, \ p \geq 2$$

improves upon the best scale equivariant estimator

$$\underline{\delta}(X) = \left(\frac{\alpha_1 - 2}{X_1}, \ldots, \frac{\alpha_p - 2}{X_p} \right)$$

under norm squared error loss.

Dasgupta (1986) also used the differential inequality approach of Berger (1980) and obtained new class of solutions to derive improvements over the best scale equivariant estimators of scale and reciprocals of scale parameters for several gamma populations. Let $(X_1, \ldots, X_p)$ be independent and X_i follows a gamma distribution with scale parameter θ_i^{-1} and shape parameter α_i, $i = 1, \ldots, p$. The estimator with the i^{th} co-ordinate

$$\delta_i(X) = \frac{\alpha_i - 2}{X_i} \left(1 + c \ (\text{sgn } m_i) X_i^{\frac{m_i}{2}} \prod_{i=1}^{p} X_i^{-\frac{m_i}{p}} \right)$$

for small positive value of c dominates the estimator with i^{th} co-ordinate $\frac{\alpha_i - 2}{X_i}$, $i = 1, \ldots, p$. The loss function considered was the sum of weighted quadratic losses. It is of the form $\sum_{i=1}^{p} c_i \theta_i^{m_i} (\frac{\delta_i}{\theta_i} - 1)^2$, where $c_i > 0$ and $m_i \neq 0$.

The estimation of common scale σ of several exponential with unequal and unknown locations has been studied by Madi and Tsui (1990) under a large class of loss functions of the form $L(\delta, \sigma) = W(\frac{\delta}{\sigma})$, where W is assumed to be continuous and strictly bowl shaped with minimum of $W(t)$ occuring at $t = 1$. The best affine equivarient estimator was shown to be inadmissible. They have provided an estimator which improves upon the best affine equivarient estimator. Since the improved estimator is not smooth, a smooth estimator was also constructed in the paper.

Elfessi and Pal (1991) have considered the estimation of the common scale parameter σ of several exponentials with unknown and unequal location parameters when censored samples are available. Let $X_{i1}, \ldots, X_{in_i}$ be the sample drawn from i^{th} population, $i = 1, \ldots, k$. Let $X_{i(1)} \leq \ldots \leq X_{i(r_i)}$ be the first r_i observations in the i^{th} sample. Define $V_i = \sum_{j=1}^{r_i} (X_{i(j)} - X_{i(1)})$, $V = \sum_{i=1}^{k} V_i$, $W = \frac{2V}{\sigma}$, $Y = \sum_{i=1}^{k} \frac{n_i X_{i(1)}}{\sigma}$ and $U = \frac{Y}{W}$. Elfessi and Pal considered scale equivariant estimators of the form $\hat{\sigma} = V\phi(U)$, and obtained improvements of the form $V\phi^*(U)$, where $\phi^*(U) = \min\left(\phi(U), \frac{1+2U}{r-k+2}\right)$, $r = \sum_{i=1}^{k} r_i$. This estimator is used to construct improved estimator of the location parameters.

Elfessi (1997) has investigated the estimation of scale parameter, its reciprocal and the location parameter in a two parameter exponential distribution based on a doubly censored sample. It was shown that the best affine equivariant estimator is inadmissible and the improving estimator is derived. The improved estimator of the receprocal of the scale parameter is similar to the one obtained by Sharma (1977). He also derived estimators of the quantile.

1.2.3 The Problem of Estimation After Selection

Let $\Pi_1, \ldots, \Pi_k$ be k populations with Π_i having an associated probability density function $f(x|\theta_i)$, $i = 1, \ldots, k$. A common problem is to select the best population or a subset of populations containing the best. The population may be termed as the best according to some characteristic such as the largest(smallest) mean, the largest(smallest) variance etc. Some typical examples are:

1. There are different types of treatments available for a particular disease. A doctor will choose the treatment which will be most effective for that treatment. Here the effectiveness may be judged by the time required to cure a patient, or the proportion of patients cured by the treatment.

2. There are a number of organic and chemical fertilizers that can be used for a certain crop. An agricultural farm owner wants to select a fertilizer for his/her crop that will give a high yield and also maintain the soil quality over a period of time.

3. An army chief will like to select the best quality guns for his army. Here the best may be decided on the basis of the proportion of successful hits or easy maneuverability of the gun carriage.

4. An industrialist will like to select that machine for his/her manufacturing facility which will give the highest production.

5. An investor will like to purchase shares of the companies which are expected to yield higher returns over next few years.

An important practical problem is to estimate the parameters of the selected population or a characteristic of the selected subset. In example 1, the doctor will like to estimate the expected time to cure or the proportion of patients cured by his/her chosen line of treatment. In example 2, the farm owner will like to have an estimate of the expected yield if he/she using the best fertilizer. An investor will like to know the estimated returns on his/her investments in the best chosen stocks. These problems are commonly referred to as "Estimation after selection".

The problem of estimation after selection differs from the classical estimation problem in a certain sense. In the usual estimation set up, the parameter to be estimated is taken to be unknown and fixed quantity. Thus an estimator T is said to be unbiased for $g(\theta)$, if $E_\theta(T) = g(\theta)$ for all θ. However, in the problem of estimation after selection, the parameter to be estimated is a random but unknown quantity, say θ_J, as it depends on the selection rule. Hence an unbiased estimator T for θ_J must satisfy $E(T-\theta_J) = 0$. Thus an unbiased estimator of θ_J is an unbiased estimator of $E(\theta_J)$ and the uniformly minimum variance unbiased estimator (UMVUE) for θ_J is, in fact, the UMVUE for $E(\theta_J)$. In the classical estimation problem, the mean squared error (MSE) of an unbiased estimator is its variance. For example, if U is unbiased for θ, then $MSE(U,\theta) = E(U-\theta)^2 = V_\theta(U)$. Therefore, if a UMVUE of θ exists, then it is uniformly better (MSE wise) than any other unbiased estimator. In contrast, the MSE of an unbiased estimator T of θ_J is,

$$MSE(T,\theta) = E(T-\theta_J)^2 = V_\theta(T) + V_\theta(\theta_J) - 2\ Cov_\theta(T,\theta_J), \tag{1.2.17}$$

where $Cov_\theta(T,\theta_J)$ denotes the covariance between T and θ_J.

The problem of estimation after selection seems to have been initially formulated and investigated by Rubinstein (1961, 1965) in the context of reliability estimation. Rubinstein used a sequential scheme for selecting the components in a manufacturing process. He derived unbiased estimators for the failure rates of the selected components. His methods also give unbiased estimator of selected Poisson parameters for a wide class of selection procedures.

The problem of estimating the mean of the selected normal population was introduced by Stein (1964). Let X_i be $N(\theta_i, 1)$, $i = 1, \ldots, k$ and be independently distributed. Suppose the population corresponding to the largest X_i is selected and M denote the mean of the selected population. For $k = 2$, Stein proved that $X_J = max(X_1, X_2)$ is admissible and minimax, when the loss function is squared error. However, it was remarked that X_J is positively biased and that its bias tends to infinity with k when θs are close. The inadmissibility of X_J was also conjectured for $k \geq 3$. Later Brown (1967) disproved this conjecture showing that X_J is admissible for any k. Practical estimators for the above problem have been proposed by Sarkadi (1967), Dahiya (1974) and Hseih (1981) (for unknown and common variance case). Cohen and Sackrowitz (1982) made an extensive study and proposed a family of estimators, which turn out to be empirical Bayes. Their main contribution is for the case $k \geq 3$. Some further work on normal populations has been done by Hwang (1987) and Venter (1988).

Major work on the problem of estimation after selection when underlying populations are non-normal was initiated by Sackrowitz and Samuel-Cahn (1984). They considered estimation of the mean of the selected exponential population when the selection rule is according to the largest observation or the smallest one. They derived the UMVUEs in both the cases. Sackrowitz and Samuel-Cahn (1987) have established some general results for the Bayes and minimax estimators for the parameters of the selected population. Vellaisamy (1993) proved some general results about uniformly minimum mean sqaured error unbiased estimation.

Estimating the mean of the selected uniform population was introduced by Vellaisamy, Kumar and Sharma (1988). Their results were further extended by Song (1992), Anand, Misra and Singh (1998). Vellaisamy and Sharma (1988) initiated the work on estimating the mean of the selecetd gamma population. Further contributions to this problem have been made by Vellaisamy and Sharma (1989), Vellaisamy (1992a), (1996) and Misra et al. (2006).

Estimation of the quantile of a selected population has been considered by Sharma and Vellaisamy (1989), Kumar and Kar (2000, 2001a,b) and Vellaisamy (2003). Mishra and van der Meulen (2001) have studied the estimation after selection in general truncation distributions. Vellaisamy and Punnen (2002) have considered the simultaneous estimation of k (where k is random) location parameters in the selected subset. The i^{th} population is exponentially distributed with common known scale parameters but unequal location parameters. They obtained some improved estimators which dominate the natural estimators under squared error loss function by solving some differential inequalities. Vellaisamy and Jain (2008) have considered the estimation of the selected parameters of a discrete exponential family. Further Vellaisamy (2008) has derived a general condition under which the unbiased estimators of the selected parameters do not exist.

1.3 A Summary of the Results in the Thesis

In **Chapter 1**, we give a detailed review of the existing literature on the following problems : (i) estimation of the reliability function, (ii) estimating the hazard rate function, (iii) estimation of reliability in stress strength models and (iv) estimation of reliability and hazard rate function from a selected component. In **Chapter 2**, some basic definitions and results of decision theory are discussed. These are useful in subsequent chapters.

In **Chapter 3**, the problem of estimation of the system reliability of a k components series system is discussed. The life time of the i^{th} component is assumed to be exponentially distributed with unknown scale λ_i^{-1}, for $i = 1, \ldots, k$. The loss taken here is the log squared error loss. The estimation of system reliability under log squared error loss is equivalent to that of estimating the sum of hazard rates under squared error loss. Independent random samples are drawn from each of k populations. Let $X_{i1}, \ldots, X_{in_i}$ be a random sample drawn from the i^{th} population, $i = 1, \ldots, k$. Then $\underline{T} = (T_1, \ldots, T_k)$ is complete and sufficient, where $T_i = \sum_{j=1}^{n_i} X_{ij}$. In Section 3.2.1, scale invariance is introduced in this problem and improvements over the existing estimators are obtained using the Brewster-Zidek (1974) technique. A generalized Bayes estimator is derived and its admissibility is established in Section 3.2.2. A numerical comparison is made to compare the risks of these improved estimators and the generalized Bayes estimator in Section 3.2.3. In Section 3.3, we consider the estimation of reliability of a series system, where the life times of individual components follow two parameter exponential distributions with a common unknown location μ and unknown but different scale parameters λ_i^{-1}, $i = 1, \ldots, k$. Let $X_{i1}, \ldots, X_{in}$ be a random sample drawn from the i^{th} population, $i = 1, \ldots, k$. $(X, \underline{T})$ is a complete and sufficient statistic for this problem, where $X = \min_i \min_j X_{ij}$, $\underline{T} = (T_1, \ldots, T_k)$ and $T_i = \frac{1}{n}(\sum_{j=1}^{n} X_{ij}) - X$. The UMVUE for the system reliability is derived in Section 3.3.2. A modified MLE in the light of Ghosh and Razmpour (1984) is proposed in Section 3.3.3. The mean squared errors of these estimators are compared through simulation in Section 3.3.4.

In **Chapter 4,** we investigate the problem of estimating the hazard rates of several exponential populations with a common unknown location μ and different scale parameters. Let the population Π_i follow a two parameter exponential distribution with a location parameter μ and scale parameter λ_i^{-1}, $i = 1, \ldots, k$. Independent random samples are drawn from each populations. Let $X_{i1}, \ldots, X_{in}$ be a random sample drawn from the i^{th} population. Further suppose $X_i = \min(X_{i1}, \ldots, X_{in})$, $X = \min(X_1, \ldots, X_k)$,

$Y_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ and $T_i = Y_i - X$, for $i = 1, \ldots, k$. Then $(X, \underline{T})$, where $\underline{T} = (T_1, \ldots, T_k)$, is complete and sufficient statistic. In Section 4.2, we study the estimation of the hazard rate of the first population when the populations have a common location. In Section 4.2.1, we obtain some basic estimators using Rao-Blackwellization. An identity is developed to derive unbiased estimators and using it, we find a UMVUE. Further the mean squared errors of these basic estimators are compared. It is shown that the modified best scale equivariant estimator δ^1_{MBS} performs the best amongst these estimators under the squared error loss. Sufficient conditions for inadmissibility are derived for affine and scale equivariant estimators in Sections 4.2.2 and 4.2.3 respectively. The results for the class of scale equivariant estimators motivated us to consider the estimation of the hazard rate of the first population when $\mu \leq 0$. The restricted MLE δ_{RML}, a restricted modified MLE δ_{RMML}, a restricted modified UMVUE δ_{RUM} and a restricted modified best scale equivariant estimators δ_{RMBS} are derived through Rao-Blackwellization in Section 4.2.4. The mean squared errors of these estimators are compared through simulation in Section 4.2.5. It is shown that δ_{RMBS} has the smallest mean squared error among these estimators. In Section 4.3, we study the simultaneous estimation of hazard rates. The loss function taken here is the sum of squared errors. Some basic estimators are proposed. In Section 4.3.1, a differential inequality is developed in the light of Berger (1980). Using this an improved estimator dominating these estimators is derived when $k \geq 2$. In Section 4.3.2, a general inadmissibility result for affine equivariant estimators is derived when the loss is the sum of squared errors or the sum of quadratic errors. In Section 4.3.3, sufficient conditions for the inadmissibility of the scale equivariant estimators are derived. In Section 4.4, the problem of estimating the common hazard rate λ of k exponential populations with different location parameters $\mu_1, \ldots, \mu_k$ is considered. Let $(X_{11}, \ldots, X_{1n}), \ldots, (X_{k1}, \ldots, X_{kn})$ be independent random samples from these populations. $(X_1, \ldots, X_k, Y)$ is a complete and sufficient statistic in this problem, where $Y = \sum_{i=1}^{k}\sum_{j=1}^{n} X_{ij}$ and $X_i = \min(X_{i1}, \ldots, X_{in})$, $i = 1, \ldots, k$. In Section 4.4.1, some basic estimators and the best affine equivariant estimators are derived using the quadratic loss. A general inadmissibility result for the class of scale equivariant estimators is derived in Section 4.4.2 using an orbit by orbit improvement technique of Brewster-Zidek (1974). It is shown that the best affine equivariant estimator δ_{BAE} of λ is in admissible. The improved estimators which dominate δ_{BAE} are obtained.

In **Chapters 5 and 6** of this thesis, we discuss the problem of estimation after selection.

In **Chapter 5,** we investigate the hazard rate estimation from a selected exponential population. Let Π_1, Π_2 be two populations where Π_i follow a one parameter exponential distribution with scale λ_i^{-1}, for $i = 1, 2$. Let $(X_{i1}, \ldots, X_{in})$ be a random sample drawn from the i^{th} population, $i = 1, \ldots, k$. The loss function is quadratic. Define $X_i = \sum_{i=1}^{k} X_{ij}$, the natural selection rule is to select the population with largest X_i, $i = 1, 2$. That is, select the population Π_i if $X_i = \max(X_1, X_2)$, $i = 1, 2$. The optimality of the natural selection rule has been investigated among others by Bahadur and Goodman (1952), Lehmann (1966) and Eaton (1967). We consider the estimation of the hazard rate from the selected component. In Section 5.2, the analogues of the MLE, the UMVUE and the best scale equivariant estimators are proposed. It is shown that these estimators are admissible within a class of estimators of the form $\frac{c}{X_M}$, where $c > 0$ and $X_M = \max(X_1, X_2)$. Minimaxity of the analogue of the best scale equivariant estimator $\delta_{N1} = \frac{n-2}{X_M}$ is proved in Section 5.2. A sufficient condition for the inadmissibility of scale equivariant estimators is derived in Section 5.4. Finally a numerical study is done to compare the risks of various estimators in Section 5.5.

In **Chapter 6,** we study the problem of estimation of the survival function of a selected exponential population. Let $\Pi_1, \ldots, \Pi_k$ be k populations where Π_i follows an exponential distribution with unknown location parameter μ_i and known common scale parameter λ^{-1}, $i = 1, \ldots, k$. Independent random samples are available from each of k populations. Let $(X_{i1}, \ldots, X_{in})$ be a random sample drawn from the i^{th} population, $i = 1, \ldots, k$. The loss function is quadratic. Let X_i denote the minimum of i^{th} sample. Then complete and sufficient statistic is $(X_1, \ldots, X_k)$. The problem of estimation of reliability function from a selected component is investigated in Section 6.2. The natural selection rule is to select the component with lowest reliability. That is, select the population Π_i, if $X_i = \min(X_1, \ldots, X_k)$. The UMVUE and some basic estimators are derived. The minimaxity of a generalized Bayes estimator is proved for $k = 2$ in Section 6.2.2. Admissible estimators within a class of estimators of the form ce^{X_J} are obtained, where $X_J = \min(X_1, X_2)$. A general inadmissibility result for the scale equivariant estimators is derived in Section 6.2.2. We study the simultaneous estimation of reliability functions in the selected subset in Section 6.3. We aim to select a subset of the populations with high reliabilities. We call the population associated with the maximum μ_i as the best population. A subset of the given k populations is selected according to Gupta's subset selection procedure (Gupta (1965)), that is, select Π_i if and only if $X_i \geq X_{(1)} - d$, for some d such that the probability of correct selection (CS) is at least P^*, a specified quantity (Gupta and Panchapakeshan (1979)), where $X_{(1)} = \max(X_1, \ldots, X_k)$. The problem is to estimate $\underline{\theta} = (\theta_1 I_1, \ldots, \theta_k I_k)$, where $\theta_i = e^{\mu_i}$,

$$\begin{aligned} I_i &= 1, \quad \text{if} \quad X_i > X_{(1)i} - d, \\ &= 0, \quad \text{otherwise}, \qquad\quad i = 1, \ldots, k, \end{aligned}$$

and

$$X_{(1)i} = \max\{X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_k\}.$$

A natural estimator and the UMVUE are derived in Section 6.3.1. For $d = 0$, sufficient conditions for inadmissibility of the estimators of the form $ce^{X_{(1)}}$ are derived. Inadmissibility of the UMVUE and the natural estimator is verified. The generalized Bayes estimator $\underline{\delta}_N^*$ of $\underline{\theta}$ with respect to the improper prior

$$\tau(\underline{\mu}) = \begin{cases} 1, & \text{if } \; -\infty < \mu_i < \infty, \; i = 1, \ldots, k, \\ 0, & \text{otherwise.} \end{cases}$$

is seen to dominate both the UMVUE and the natural estimator under squared error loss. In Section 6.3.3, some improved estimators which dominate the estimators of the form $\underline{\delta}^c = \{ce^{X_1} I_1, \ldots, ce^{X_k} I_k\}$, where $0 < c < 1$, are constructed using a differential inequality approach used by Vellaisamy (1992) and Vellaisamy and Punnen (2002). In Section 6.4, we study the problem of estimation of reliability function of the selected negative exponential population. The UMVUE is derived in Section 6.4.1. Finally a numerical comparison is made between the UMVUE and the natural estimators in Section in 6.4.2.