CHAPTER I

INTRODUCTION

1.1. Magneto-hydrodynamic principles and application to problems

The subject - Magneto-hydrodynamics - deals with the flow of an electrically conducting fluid in the presence of an electromagnetic field. An induced electric current is generated due to the flow of the conducting fluid in the presence of a magnetic field which interacts with the latter and produces a ponderomotive force called Lorentz-force. This force modifies the flow field while the electromagnetic field itself is modified by the induced electric current. Hartmann and Lazarus (1937) initiated the study of the subject in the name of Hg-Dynamics in their effort to pump mercury by the exploitation of the hydrodynamical and electromagnetic fields. However, systematic studies under the present name began with the discovery of transverse waves by Alfvén (1942,1943) while he was engaged in the theoretical investigations for exploring the theory of sunspots.

Although the study of the subject started nearly three decades ago it has found useful applications to various types of problems. Its application in Astrophysical problems has been made by many authors. Explanations of the occurence of the sun-spots due to the presence of a magnetic core inside the sun has been attempted by Larmor (1934), Walen (1944), Cowling (1945), Menzel (1951), Dungey (1953) and others. The oscillation theory of variable stars has been given by Cowling (1946). In geophysical problems, the maintainance of the earth's magnetic field and its secular variation have been studied by Bullard (1948,1949), Elsasser (1950,1956), Parker (1955), Hide (1965) and others. The application to engineering and technical problems has been reviewed by Kármán (1959). The magneto-hydrodynamic (MHD) power generation has been studied extensively in theory, design and experiments and are reported in papers and reviews by Sutton (1959), Curzon et al (1960), Mannel and Mather (1962), Mcgrath et al (1963) and many others. Post (1956), Bishop (1958), Kuchatov (1956) and Spitzer (1956) have studied the use of MHD principles in controlled fusion research. Its application to lubrication problems have been studied by Huges and Elco (1962), Snyder (1962), Agarwal (1963), Ramanaiah (1966) and many others.

1.2. Fundamental Equations of Magneto-gasdynamics

Let us consider the flow of an electrically conducting fluid in the presence of an electromagnetic field. The fundamental equations governing the flow are the equations of fluid dynamics together with the electromagnetic field equations.

The equations of fluid dynamics consist of the equations for the conservation of mass, momentum and energy, respectively known as the equation of continuity, the equation of motion and the energy equation. They are:

(i) The equation of continuity,

$$\frac{\partial \rho}{\partial t} + div\,(\rho \bar{q}) = 0\,, \tag{1.1}$$

(ii) The equation of motion,

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot grad)\bar{q}\right] = \rho\left[\bar{F} + \bar{F}_m\right] + \nabla\cdot P\,, \tag{1.2}$$

(iii) The energy equation,

$$\rho T\left[\frac{\partial S}{\partial t} + (\bar{q}\cdot grad)S\right] = Q_J - div\,Q_c\,, \tag{1.3}$$

where $\bar{q}$ is the velocity field, ρ the density, T the temperature and S the entropy per unit mass of the fluid,

$\bar{F}$ is the mechanical body force, $\bar{F}_m$ the body force due to the electro-magnetic field, P is the stress dyad with components p_{ij}, Q_g is the heat generated per unit volume and Q_c is the heat flow. The physical state of the fluid is given by a single relation connecting p, ρ, T known as the equation of state,

$$f(p, \rho, T) = 0 , \tag{1.4}$$

where p is known as the fluid pressure.

The Maxwell's electro-magnetic field equations in rationalized m.k.s. units are

$$\text{curl}\ \bar{H} = \bar{J} , \tag{1.5}$$

$$\text{curl}\ \bar{E} = -\frac{\partial B}{\partial t} , \tag{1.6}$$

$$\text{div}\ \bar{B} = 0 , \tag{1.7}$$

$$\text{div}\ \bar{D} = \rho_e , \tag{1.8}$$

$$\bar{D} = \epsilon \bar{E} , \quad \bar{B} = \mu_e \bar{H} , \tag{1.9}$$

where $\bar{E}$ is the electric field, $\bar{D}$ the electric displacement vector, $\bar{H}$ the magnetic field, $\bar{B}$ the magnetic induction, $\bar{J}$ the electric current density, and ϵ, μ_e and ρ_e are respectively the dielectric constant, magnetic permeability and charge density of the fluid. The current density $\bar{J}$ is given by Ohm's law,

$$\bar{J} = \rho_e \bar{q} + \bar{j} + \frac{\partial \bar{D}}{\partial t} , \tag{1.10}$$

with $\bar{j} = \sigma(\bar{E} + \bar{q} \times \bar{B})$, σ being the electrical conductivity of the fluid. The first term in (1.10) is the convection current, the second term the conduction current and the third is the displacement current. The Lorentz force $\bar{F}_m$ is given by

$$\rho \bar{F}_m = \rho_e \bar{E} + \bar{J} \times \bar{B} . \tag{1.11}$$

The problem of magneto-fluid dynamics is to find the velocity vector $\bar{q}$ together with any two of the thermodynamic variables p, ρ and T , and the electromagnetic field $\bar{E}$ and $\bar{H}$. In order to solve equations (1.1) to (1.3) we must know $\bar{F}$, $\bar{F}_m$, p_{ik} , Q_J and Q_c . On the other hand, to determine the electro-magnetic field from the Maxwell's equations we have to know the fluid-velocity $\bar{q}$. Thus the flow field and the electro-magnetic field are closely interconnected.

For Newtonian viscous fluids, the stress-strain relation is given by,

$$p_{ik} = -p\delta_{ik} + \tau_{ik} , \tag{1.12}$$

$$\tau_{ik} = \mu\left(-\tfrac{2}{3}\theta\,\delta_{ik} + e_{ik}\right). \tag{1.13}$$

Here μ is the coefficient of viscosity, δ_{ik} is the Kronecker delta and e_{ik} is the rate of strain tensor given by

$$e_{ik} = v_{i,k} + v_{k,i}, \tag{1.14}$$

and

$$\Theta = v_{i,i},$$

where a comma denotes a covariant differentiation with respect to the letter following it.

The equation (1.2) reduces to

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot \mathrm{grad})\bar{q}\right] = \rho\bar{F} + \nabla P + \rho_e\bar{E} + \bar{J}\times\bar{B}, \tag{1.15}$$

As regards the energy-equation (1.3), the heat generation Q_g is due to the viscous dissipation and the Joule heating, so that

$$Q_g = \Phi + \frac{\bar{J}^2}{\sigma}, \tag{1.16}$$

$$\Phi = \tfrac{1}{2}\tau_{ik}e_{ik} = \tfrac{1}{2}\mu\left(e_{ik}e_{ik} - \tfrac{4}{3}\theta^2\right). \tag{1.17}$$

If the temperature is not too high (so that the heatradiation is negligible) the heat flow Q_c may be supposed to be due to conduction only. Thus we have

$$Q_c = -k \, \mathrm{grad}\, T \,, \tag{1.18}$$

where k is the thermal conductivity of the fluid. The energy equation (1.3) can then be written as

$$\rho c_v \left[\frac{\partial T}{\partial t} + (\bar{q} \cdot \mathrm{grad}) T \right] = k \nabla^2 T + \Phi + \frac{\bar{J}^2}{\sigma} \,, \tag{1.19}$$

where c_v is the specific heat of the fluid at constant volume.

1.3. Magneto-hydrodynamic Equations

The general equations of magneto-gasdynamics, given in the previous section, are complicated and it is possible to introduce a few simplifying assumptions without sacrificing the essential features of the problem. In most of the Magneto-hydrodynamic problems the displacement current is ignored. This is justified when the fluid-velocity is negligibly small in comparison to the velocity of light [see Bullard (1955)]. Then

$$\mathrm{curl}\, \bar{H} = \bar{j} + \rho_e \bar{q} \tag{1.20}$$

For fluids which are almost neutral, $\rho_e \simeq 0$, and the convection current $\rho_e \bar{q}$ is negligible in comparison to the conduction current $\bar{j}$ [see Bullard (1955)]. The equation (1.10) can then be written as

$$\bar{J} = \sigma(\bar{E} + \bar{q} \times \bar{B}) \quad . \tag{1.21}$$

It may be noted that the assumption of exact neutrality would lead to the superfluous condition, $div\ D = 0$, by equation (1.8). Substituting $\bar{E}$ from (1.6) into (1.5) and using (1.21) we have,

$$\frac{\partial \bar{B}}{\partial t} = curl(\bar{q} \times \bar{B}) - curl\left[\frac{1}{\sigma} curl \frac{\bar{B}}{\mu}\right]. \tag{1.22}$$

This is the equation of induction and can rewritten as

$$\frac{\partial \bar{B}}{\partial t} = curl(\bar{q} \times \bar{B}) + \eta \nabla^2 \bar{B} \tag{1.23}$$

where $\eta = (1/\sigma \mu_e)$ is called the magnetic diffusivity or magnetic viscosity.

When the fluid is incompressible the equations (1.1) and (1.15) can be rewritten as

$$div\ \bar{q} = 0 \ , \tag{1.24}$$

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot grad)\bar{q}\right] = \rho \bar{F} - grad\, p + \mu \nabla^2 \bar{q} + \bar{J} \times \bar{B} . \tag{1.25}$$

The fundamental equations of magneto-hydrodynamics for an incompressible, viscous electrically conducting fluid are the equations (1.7), (1.19) and (1.23) - (1.25). Considering the components of the vector quantities the total number of equations they give rise to is nine. The unknown quantities to be sought are the three components of each of both the velocity vector $\bar{q}$ and the magnetic induction vector $\bar{B}$, the fluid-pressure p and the temperature T. Hence we have eight unknown quantities to solve. One of the component equations of (1.23) can be derived from the other two with the help of equation (1.7) and thus we have only eight independent equations to solve the eight unknowns and the problem is determinate. The electric field and the current density can then be determined from (1.6) and (1.21). The magneto-hydrodynamic interaction is evident through the interaction terms 'curl $(\bar{q} \times \bar{B})$' in equation (1.23) and '$\bar{J} \times \bar{B}$' in equation (1.25).

1.4. Boundary Conditions

The flow field and the electro-magnetic field are to be determined by solving the fundamental equations stated in the previous section under appropriate boundary conditions for the flow filled and the electro-magnetic field.

(i) Boundary conditions for the flow field:

They are the same as in ordinary hydrodynamics. For example, in the case of a solid body moving through a viscous fluid, the velocity of a fluid particle at any point on the surface of the body relative to the solid body should vanish.

(ii) Boundary conditions for the electro-magnetic field:

The electro-magnetic field, in general, suffers from an abrupt change at a surface where the properties of the medium are discontinuous. Integrating the equations (1.5) and (1.6) over an infinitesimal area containing a line element ds of the discontinuity surface S gives respectively

$$\{E_t\} = 0 \quad , \tag{1.26}$$

$$\{H_t\} = \bar{j}_s \times \bar{n} \quad . \tag{1.27}$$

Here E_t and H_t are the components of $\bar{E}$ and $\bar{H}$ parallel to the surface of discontinuity S, and $\bar{j}_s$ is the current density on it. The notation $\{Q\}$ signifies,

$$\{Q\} = Q_+ - Q_- \quad , \tag{1.28}$$

namely, the jump in the value of Q when one crosses the surface S from negative side to the positive side and $\bar{n}$ is the unit vector along the normal to the surface in the positive direction. Similarly, integration of (1.7) and (1.8) over an infinitesimal volume enclosing a surface element dS of S gives respectively,

$$\{D_n\} = \rho_s \quad , \quad \{B_n\} = 0 \quad , \tag{1.29}$$

where D_n and B_n are the normal components of $\bar{D}$ and $\bar{B}$ respectively and ρ_s is the surface density of charge over S .

Thus, when there is no surface-current and surface-charge on the surface of discontinuity we have the following boundary conditions:

(a) The tangential components of $\bar{H}$ and $\bar{E}$ are continuous across the surface.

(b) The normal components of $\bar{D}$ and $\bar{B}$ are continuous across the surface.

1.5. Magneto-hydrodynamic equations in spherical and cylindrical polar co-ordinates

Writing (u, v, w) as the components of velocity and $(H_r, H_\theta, H_\varphi)$ as the components of the magnetic field along the directions of (r, θ, φ) respectively, the magneto-hydrodynamic <u>equations in spherical polar co-ordinates</u> are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \varphi} - \frac{v^2 + w^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$+ \nu\left[\nabla^2 u - \frac{2u}{r^2} - \frac{2v}{r^2}\cot\theta - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial w}{\partial \varphi}\right]$$

$$+ \frac{\mu_e}{\rho}\left[\frac{H_\varphi}{r}\left\{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \varphi} - \frac{\partial}{\partial r}(r H_\varphi)\right\} - \frac{H_\theta}{r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial \theta}\right\}\right], \quad (1.30)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta = -\frac{1}{\rho}\frac{\partial p}{r\,\partial \theta}$$

$$+ \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2\sin^2\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial w}{\partial \varphi}\right]$$

$$+ \frac{\mu_e}{\rho}\left[\frac{H_r}{r}\left\{\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial \theta}\right\} - \frac{H_\varphi}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(H_\varphi \sin\theta) - \frac{\partial H_\theta}{\partial \varphi}\right\}\right],$$

$$(1.31)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{uw}{r} + \frac{vw}{r}\cot\theta$$

$$= -\frac{1}{\rho}\frac{\partial p}{r\sin\theta\,\partial \varphi} + \nu\left[\nabla^2 w + \frac{2}{r^2\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v}{\partial \varphi}\right.$$

$$\left. - \frac{w}{r^2\sin^2\theta}\right] + \frac{\mu_e}{\rho}\left[\frac{H_\theta}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(H_\varphi \sin\theta) - \frac{\partial H_\theta}{\partial \varphi}\right\}\right.$$

$$\left. - H_r\left\{\frac{1}{r\sin\theta}\frac{\partial H_\theta}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi)\right\}\right], \quad (1.32)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(u r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} = 0, \tag{1.33}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(H_r r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial H_\varphi}{\partial\varphi} = 0 \tag{1.34}$$

$$\frac{\partial H_r}{\partial t} + u\frac{\partial H_r}{\partial r} + \frac{v}{r}\frac{\partial H_r}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial H_r}{\partial\varphi} - H_r\frac{\partial u}{\partial r}$$

$$- \frac{H_\theta}{r}\frac{\partial u}{\partial\theta} - \frac{H_\varphi}{r\sin\theta}\frac{\partial u}{\partial\varphi} = \eta\Big\{\nabla^2 H_r - \frac{2H_r}{r^2}$$

$$- \frac{2H_\theta}{r}\cot\theta - \frac{2}{r^2}\frac{\partial H_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial H_\varphi}{\partial\varphi}\Big\}, \tag{1.35}$$

$$\frac{\partial H_\theta}{\partial t} + u\frac{\partial H_\theta}{\partial r} + \frac{v}{r}\frac{\partial H_\theta}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial H_\theta}{\partial\varphi} + \frac{uH_\theta}{r}$$

$$- H_r\frac{\partial v}{\partial r} - \frac{H_\theta}{r}\frac{\partial v}{\partial\theta} - \frac{H_\varphi}{r\sin\theta}\frac{\partial v}{\partial\varphi} - \frac{vH_r}{r}$$

$$= \eta\Big[\nabla^2 H_\theta + \frac{2}{r^2}\frac{\partial H_r}{\partial\theta} - \frac{H_\theta}{r^2\sin^2\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial H_\varphi}{\partial\varphi}\Big], \tag{1.36}$$

$$\frac{\partial H_\varphi}{\partial t} + u\frac{\partial H_\varphi}{\partial r} + \frac{v}{r}\frac{\partial H_\varphi}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial H_\varphi}{\partial\varphi} + \frac{uH_\varphi}{r}$$

$$+ \frac{vH_\varphi}{r}\cot\theta - H_r\frac{\partial w}{\partial r} - \frac{H_\theta}{r}\frac{\partial w}{\partial\theta} - \frac{H_\varphi}{r\sin\theta}\frac{\partial w}{\partial\varphi}$$

$$- \frac{H_r w}{r} - \frac{H_\theta w}{r}\cot\theta = \eta\Big[\nabla^2 H_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial H_r}{\partial\varphi}$$

$$+ \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial H_\theta}{\partial\varphi} - \frac{H_\varphi}{r^2\sin^2\theta}\Big], \tag{1.37}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{1.38}$$

One of the three equations (1.35) - (1.37) is redundant. The energy equation in this co-ordinate system is

$$\begin{aligned}\rho c_v\left[\frac{\partial T}{\partial r} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial T}{\partial\varphi}\right] &= k\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right. \\ &+ \left.\frac{\cot\theta}{r^2}\frac{\partial T}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}\right] + 2\mu\left[\left(\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}\right)^2\right. \\ &+ \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{r\sin\theta\,\partial\varphi} + \frac{u}{r} + \frac{v}{r}\cot\theta\right)^2 + \\ &+ \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\varphi} - \frac{w}{r}\cot\theta\right)^2 + \frac{1}{2}\left(\frac{\partial u}{r\partial\theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right)^2 \\ &+ \left.\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} - \frac{w}{r} + \frac{\partial w}{\partial r}\right)^2\right] + \frac{1}{\sigma}\left[\frac{1}{r^2\sin^2\theta}\left\{\frac{\partial}{\partial\theta}(H_\varphi\sin\theta)\right.\right. \\ &- \left.\frac{\partial H_\theta}{\partial\varphi}\right\}^2 + \left\{\frac{1}{r\sin\theta}\frac{\partial H_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi)\right\}^2 \\ &+ \left.\left\{\frac{1}{r}\frac{\partial}{\partial r}(r H_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial\theta}\right\}^2\right].\end{aligned} \tag{1.39}$$

The field equations in <u>cylindrical polar co-ordinates</u> are

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2}\right. \\ &- \left.\frac{2}{r^2}\frac{\partial v}{\partial\theta}\right] + \frac{\mu_e}{\rho}\left[H_z\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) - \frac{H_\theta}{r}\left\{\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial\theta}\right\}\right],\end{aligned} \tag{1.40}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\cdot\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left\{\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right\} + \frac{\mu_e}{\rho}\left[\frac{H_r}{r}\left(\frac{\partial H_\theta}{\partial r} - \frac{\partial H_r}{\partial \theta}\right) - \frac{H_z}{r}\left\{\frac{\partial H_z}{\partial \theta} - \frac{\partial}{\partial z}(r H_\theta)\right\}\right], \tag{1.41}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \frac{\mu_e}{\rho}\left[\frac{H_\theta}{r}\left\{\frac{\partial H_z}{\partial \theta} - \frac{\partial}{\partial z}(r H_\theta)\right\} - H_r\left\{\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right\}\right], \tag{1.42}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(u r) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{1.43}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(H_r r) + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} + \frac{\partial H_z}{\partial z} = 0, \tag{1.44}$$

$$\frac{\partial H_r}{\partial t} + u\frac{\partial H_r}{\partial r} + \frac{v}{r}\frac{\partial H_r}{\partial \theta} + w\frac{\partial H_r}{\partial z} - H_r\frac{\partial u}{\partial r} - \frac{H_\theta}{r}\frac{\partial u}{\partial \theta} - H_z\frac{\partial u}{\partial z} = \eta\left[\nabla^2 H_r - \frac{H_r}{r^2} - \frac{2}{r^2}\frac{\partial H_\theta}{\partial \theta}\right], \tag{1.45}$$

$$\frac{\partial H_\theta}{\partial t} + u\frac{\partial H_\theta}{\partial r} + \frac{v}{r}\frac{\partial H_\theta}{\partial \theta} + w\frac{\partial H_\theta}{\partial z} + \frac{u H_\theta}{r} - H_r\frac{\partial v}{\partial r} - \frac{H_\theta}{r}\frac{\partial v}{\partial \theta} - H_z\frac{\partial v}{\partial z} - H_r\frac{v}{r} = \eta\left[\nabla^2 H_\theta - \frac{H_\theta}{r^2} + \frac{2}{r^2}\frac{\partial H_r}{\partial \theta}\right], \tag{1.46}$$

$$\frac{\partial H_z}{\partial t} + u\frac{\partial H_z}{\partial r} + \frac{v}{r}\frac{\partial H_z}{\partial \theta} + w\frac{\partial H_z}{\partial z} - H_r\frac{\partial w}{\partial r}$$

$$- \frac{H_\theta}{r}\frac{\partial w}{\partial \theta} - H_z\frac{\partial w}{\partial z} = \eta \nabla^2 H_z \, , \tag{1.47}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \, , \tag{1.48}$$

(u, v, w) are the components of velocity and (H_r, H_θ, H_z) the components of the magnetic field along the directions of (r, θ, z) respectively. Here, again, one of the equations (1.45) - (1.47) is redundant.

The <u>energy equation</u> in this system of co-ordinates is

$$\rho C_v\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z}\right] = k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right.$$

$$\left. + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + 2\mu\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left\{\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{u}{r}\right\}^2\right.$$

$$+ \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{r\,\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2$$

$$\left. + \frac{1}{2}\left(\frac{\partial w}{r\,\partial \theta} + \frac{\partial v}{\partial z}\right)^2\right] + \frac{1}{\sigma}\left[\frac{1}{r^2}\left\{\frac{\partial H_z}{\partial \theta} - \frac{\partial}{\partial z}(r H_\theta)\right\}^2\right.$$

$$\left. + \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right)^2 + \frac{1}{r^2}\left\{\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial \theta}\right\}^2\right] . \tag{1.49}$$