CHAPTER I

INTRODUCTION

Every solution of the equations of hydrodynamics or of hydromagnetics may not be realizable in practice. This is due to the fact that the flow may be unstable to infinitesimal or finite amplitude disturbances. Linear stability theory deals with the question of stability to infinitesimal disturbances, while the question of stability to finite amplitude disturbances falls in the domain of nonlinear stability theory.

The hydrodynamic or hydromagnetic system, whose stability is being studied, can be described by a set of non-dimensional parameters. The stability of the system depends on the numerical values of these parameters. The parameter space can be divided into stable and unstable zones. In the stable zone, any arbitrary disturbance to the system must tend asymptotically to zero as time tends to infinity. In the unstable zone there must be at least one particular mode of disturbance which grows with time. The locus of points which separates the stable zone(s) from the unstable zone(s) defines the states of marginal stability of the system. States of marginal stability (or marginal states) can be of two types. In the first type, transition from stability to instability takes place via a marginal state exhibiting a stationary pattern of motions. In this case the marginal state is said to be stationary and the principle of exchange of stabilities is said to be valid. In the second type, the transition from stability to instability takes place via a marginal state exhibiting oscillatory motion with a certain definite characteristic frequency. In this case, referred to as overstability (Chandrasekhar [1]), a slight displacement provokes restoring forces so strong as to overshoot the corresponding position on the other side of equilibrium. However, this classification of the marginal states is valid for dissipative systems.

The total volume of scientific literature dealing with linear hydrodynamic and hydromagnetic stability is considerable. Hence the discussion which follows is restricted to only those papers which have direct relevance to the problems investigated in this thesis.

(a) CENTRIFUGAL INSTABILITY

The stability of cylindrical Couette flow of an incompressible inviscid fluid was first investigated by Rayleigh [2] who showed that a stratification of angular momentum about an axis is stable if and only if it increases monotonically outward. This necessary and sufficient condition for stability is known as Rayleigh's criterion. The viscous stability of two-dimensional stagnation-point flow was investigated by Görtler [3] and Hämmerlin [4] who showed that the flow is unstable. This result may be related to Rayleigh's problem because in this case the circulation (product of local velocity and the local radius of curvature) increases as the local centre of curvature is approached normal to the curved streamlines.

(b) THERMAL INSTABILITY

The problem of the onset of thermal instability (referred to as Bénard convection) in a horizontal layer of fluid, heated from below in a field of gravity was examined theoretically by Rayleigh [5]. His analysis was confined to the case of two free boundaries. He showed that the stability of the layer is determined by the numerical value of a non-dimensional parameter, referred to as Rayleigh number, which depends on the temperature gradient, the depth of the layer, the acceleration due to gravity and on the coefficient of thermal expansion, the thermal diffusivity and the kinematic viscosity of the fluid. If the Rayleigh number is below a certain value, the layer is stable. Instability sets in as soon as the Rayleigh number exceeds this critical value. He also proved that the marginal state is stationary. Pellew and Southwell [6] extended Rayleigh's analysis to include other types of boundary conditions and proved that even in these cases the principle of exchange of stabilities is valid. They also obtained a variational principle for this problem and used it for approximate calculation of the critical Rayleigh number. Chandrasekhar [7] and Chandrasekhar and Elbert [8] extended Rayleigh's problem to include the effect of rigid rotation. They obtained a variational principle for this problem and calculated approximate values of the critical Rayleigh number as a function of the Taylor number (a non-dimensional parameter which gives a measure of the rotation rate). They showed that in this case the marginal state may be oscillatory particularly for fluids with low Prandtl number.

The problem of thermal instability in spheres and in spherical shells, heated within, was examined by Chandrasekhar [9, 10]. Bisshopp [11] analysed the problem for spheres, heated within and subject to rigid rotation about a diameter as the axis. His analysis, however, was confined to axisymmetric disturbances. The analysis for non-axisymmetric disturbances was carried out by Roberts [12] who showed that except for the smallest Taylor numbers, instability first sets in as a non-axisymmetric disturbance.

Horton and Rogers [13] and Lapwood [14] considered the problem of thermal instability in a horizontal layer of fluid in a porous medium, due to a uniform adverse temperature gradient. The problem of thermal instability in fluid-filled porous spheres heated within was examined by Pradhan and Patra [15].

Hurle, Jakeman and Pike [16] made the analysis of Pellow and Southwell [6] more realistic by considering, for the case of a fluid layer bounded by two rigid surfaces, that the media bounding the fluid have a finite thermal conductivity, so that the bounding surfaces are no longer at constant temperature as had been assumed by Pellow and Southwell [6]. In fact the assumption of constant temperatures at the boundaries is valid when their conductivities are very large.

(c) THERMOCONVECTIVE WAVES

Transverse thermal waves in an isothermal viscous fluid are known to be strongly damped even in the absence of gravity forces (Carslaw and Jaeger [17]). It was first shown by Luikov and Berkovsky [18] that in a viscous liquid heated from below, such that Bénard convection does not set in, transverse plane waves may propagate almost undamped under certain conditions. These waves are known as thermoconvective waves. Subsequently, undamped propagation of thermoconvective waves was shown to be possible in the presence of a magnetic field by

Takashima [19] only under astrophysical conditions and in certain viscoelastic fluids by Gupta and Gupta [20]. Kock and Schneider [21] studied the effect of compressibility on the propagation of thermoconvective waves. Moreover, they considered a horizontally bounded layer and showed that the horizontal walls do not prevent the existence of weakly damped thermoconvective waves provided that the Rayleigh number is large.

(d) HYDROMAGNETIC STABILITY OF NON-DISSIPATIVE FLOWS

In a theoretical study of the inviscid stability of parallel shear flow of a density-stratified fluid, Miles [22] and Howard [23] derived a sufficient condition for stability of the flow in a gravity field. They showed that if a certain non-dimensional parameter, known as the Richardson number based on the density-gradient and the shear in the parallel flow everywhere exceeds 1/4, then the flow is stable to infinitesimal disturbances. Howard [23] also derived a semicircle theorem which demonstrates that the complex wave velocity of any unstable mode must lie in a certain semicircle in the complex plane. Howard and Gupta [24] derived Richardson number type criteria and circle theorems for a number of cases of non-dissipative hydrodynamic and hydromagnetic helical flows. However, in most of their analysis, only axisymmetric disturbances were considered. Barston [25] gave a general method of constructing circle theorems which localize the complex eigenfrequencies arising in the linear stability analysis of conservative steady flows. As applications, he considered helical flow of an inviscid, incompressible fluid and rotating flow of an inviscid, incompressible, perfectly conducting fluid permeated by an axial magnetic field. In his analysis axisymmetric as well as non-axisymmetric disturbances were considered. Using Barston's technique Ganguly and Gupta [26] obtained a number of results for stability of non-dissipative hydromagnetic helical flows to all normal modes of disturbances. Barston's technique, however, yields a circle theorem with the radius of the circle, in general, dependent on the wavenumbers.

The effect of a radial magnetic field on the stability of viscous flow between two rotating electrically non-conducting cylinders was studied by Antimirov and Kolyshkin [27]. They calculated the values of critical Taylor number as a function of wavenumber for different values of the Hartmann number.

The stability of combined radial and rotational flow (spiral flow) of an inviscid fluid was studied by Hazlehurst [28]. Restricting his analysis to axisymmetric disturbances he found that the flow is unstable or stable according as it is directed towards or away from the axis of symmetry. Bahl [29] examined the stability of viscous flow between two rotating porous concentric cylinders. While studying the effects of suction on critical Taylor number, he found that injection at the outer cylinder improves stability but suction has an opposite effect. Chang and Sartory [30] examined the stability of the flow of a viscous electrically-conducting fluid between two concentric rotating porous cylinders in the presence of an axial magnetic field both when the marginal state is stationary or oscillatory.

Gilman [31] analysed the stability of a horizontal layer of a perfectly conducting inviscid gas in the presence of a horizontal magnetic field. He showed that when the strength of the magnetic field and the fluid density are functions of the vertical coordinate, instability sets in by a mechanism known as magnetic bouyancy if the basic magnetic field strength decreases with height faster than the density. He also examined the effect of rigid rotation on the stability of the system. Acheson and Gibbons [32] extended Gilman's analysis to the case of a uniformly rotating gas in a cylindrical geometry. However, in both these investigations infinite thermal conductivity of the fluid was assumed. In an exhaustive study Acheson [33] considered the stability of a rotating spherical body of compressible fluid permeated by a toroidal magnetic field taking into account all diffusive phenomena (viscous, thermal and ohmic). He carried out local stability analyses for low frequency disturbances. Gibbons [34] examined the validity of local stability analysis for a particular problem involving instability due to magnetic buoyancy in a rotating plane layer of compressible fluid. However, he retained the low frequency approximation in his analysis.

(e) HYDROMAGNETIC STABILITY OF DISSIPATIVE FLOWS

Taylor [35] extended the analysis of Rayleigh [2] to include the effect of viscosity and showed that viscosity has a stabilizing influence on the flow. He showed that, in this case, the stability of the flow is determined by the numerical value of a non-dimensional parameter, referred to as Taylor number, which gives a measure of the extent to which Rayleigh's criterion is violated. He calculated, under certain simplifying assumptions, the numerical value of the critical Taylor number at which instability first sets in. The effect of a uniform axial magnetic field on the stability of cylindrical Couette flow of a viscous electrically-conducting fluid was studied by Chandrasekhar [36]. He found that the magnetic field tends to stabilize the flow. Chandrasekhar [36] restricted his analysis to the case of a weakly conducting fluid (low magnetic Prandtl number approximation). Kurzweg [37] extended his analysis by considering fluids of arbitrary electrical conductivity.

Theoretical as well as experimental work has been done on the stability of hydrodynamic as well as hydromagnetic flows in which the basic state is oscillatory. The stability of viscous cylindrical Couette flow, when the rotation rate of the inner cylinder is modulated sinusoidally, was investigated experimentally by Donnelly [38]. He showed that the onset of instability can be inhibited by modulation. The theoretical analysis for this problem was carried out by Riley and Laurence [39]. The effect of an oscillatory magnetic field on the stability of parallel flow was examined by Drazin [40]. He carried out detailed stability analysis for an inviscid, perfectly conducting fluid. Instability of a linear pinch in an oscillating magnetic field was studied by Tayler [41].

In the light of the above literature survey the following problems have been investigated.

In chapter II the linear stability of steady two-dimensional boundary layer flow of an incompressible viscous fluid over a flat deformable sheet is investigated when the sheet is stretched in its own plane with an outward velocity proportional to the distance from a point on it, the analysis being confined to infinitesimal, non-propagating, Görtler type disturbances. The flow is shown to be stable. The significance of this result in polymer processing applications is pointed out.

In chapter III the manner of the onset of thermal instability in fluid-filled porous spheres and spherical shells heated from within is investigated. The first part of this chapter examines the effect of rigid rotation of the system on the onset of instability. A general disturbance is analysed into modes in terms of spherical harmonics and the criterion for the onset of thermal instability is determined for the various modes. From the numerical results conclusions are drawn about the mode for which instability first sets in. The critical Darcy-Rayleigh number is obtained as a function of the Darcy-Taylor number. It is also found that for the most unstable mode, the marginal state is characterized by a very slow westward rotation of the entire convection pattern around the globe. The second part of this chapter examines the effect of a finite thermal conductivity of the core on the onset of thermal instability in a fluid-filled porous spherical shell. It is shown that as the conductivity of the core decreases, the critical Darcy-Rayleigh number decreases and the most unstable mode tends to shift towards a lower wavenumber associated with the polar angle in a system of spherical coordinates. The problem of the onset of thermal instability in fluid-filled porous spheres and spherical shells is of relevance in connection with problems of geophysical and astrophysical interest, e.g., convection in the earth's mantle.

In chapter IV the propagation of transverse thermal waves in a layer of binary mixture in the presence of uniform thermal and concentration gradient is studied. It is shown that when the thermal diffusivity K_T is equal to the mass diffusivity K_S, undamped waves do not exist, but when $K_T > K_S$, these waves may exist provided the layer is heated from below (such that thermohaline instability does not occur) and the concentration of the solute in the mixture decreases vertically upwards. The analysis also reveals the interesting result that when $K_T < K_S$, undamped waves can propagate even if the layer is heated from above provided the concentration increases vertically upwards. It is further shown that as in a homogeneous fluid, weakly damped low frequency waves can propagate and the regions in the parameter space where such propagation is possible are delineated. High frequency waves are shown to be always strongly damped. This problem may have some relevance to oceanography e.g. the stratification of salt in oceans might affect the propagation of thermoconvective waves in the presence of variations in solar heating.

In the first part of chapter V the linear stability of the flow of an incompressible, inviscid, perfectly conducting fluid between two non-porous coaxial cylinders having an axial and an azimuthal component of velocity and permeated by a magnetic field having radial, azimuthal and axial components is examined. It is shown that in such a flow known as helical flow, the presence of a radial component of the magnetic field imposes certain restrictions on the basic state. It is further shown that for any basic state consistent with the governing equations, the flow is stable even in the presence of non-axisymmetric disturbances. In the second part of this chapter, the linear stability of the radial flow of an incompressible, inviscid, perfectly conducting fluid between two coaxial porous cylinders permeated by a radial magnetic field is examined. It is found that the ratio of the local Alfvén velocity and the flow velocity in the basic state must be constant over the entire flow field. It turns out that if the Alfvén velocity exceeds the flow velocity, then the flow is stable. The problem considered in the first part of this chapter has a bearing on MHD power generation and such a configuration occurs in magnetohydrodynamically driven vortices. Radial magnetic field also finds application in the study of magnetic flux generation in the interior of planets and stars.

In chapter VI the linear stability of non-dissipative hydromagnetic flow of an incompressible density-stratified fluid between two coaxial cylinders to non-axisymmetric disturbances is studied. A sufficient condition for stability is derived for rigidly rotating fluid permeated by an azimuthal magnetic field against all disturbances. In the case of helical flow permeated by an axial magnetic field, a bound on the growth rate independent of wavenumbers is obtained.

In chapter VII the hydromagnetic stability of a rotating compressible fluid is studied. In the first part of this chapter the linear stability of a plane layer of a perfectly conducting gas permeated by a vertically stratified horizontal magnetic field is examined when the layer rotates about a horizontal axis perpendicular to the magnetic field. Using WKBJ approximation, it is shown that under adiabatic conditions the layer would be unstable to perturbations whose wavelength along the magnetic field is much larger than a typical wavelength along a horizontal direction provided the destabilizing influence due to magnetic buoyancy overcomes the stabilizing influence of a positive entropy stratification. In the second part of this chapter the stability of a cylindrical rotating mass of a perfectly conducting inviscid gas in the presence of a toroidal magnetic field is studied. It is shown that under adiabatic conditions, unstable low frequency waves propagate along the basic rotation outside a critical radius depending on the basic acoustic speed. Inside the critical radius, however, unstable waves propagate against the basic rotation. The hydromagnetic stability of a rotating compressible fluid is of importance in several geophysical and astrophysical problems concerning planetary and stellar atmospheres.

In chapter VIII the stability of cylindrical Couette flow of an electrically conducting incompressible viscous fluid in the presence of a modulated axial magnetic field is examined. With the usual simplifying assumptions of low magnetic Prandtl number and narrow gap approximation, it is shown that for low frequencies and for low values of the Chandrasekhar number, increase in the amplitude of modulation tends to stabilize the system.

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