STUDIES ON TRIANGLES AND TETRAHEDRA IN PROJECTIVE SPACE

A THESIS

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PREFACE

The origin of any branch of science can be traced far back in human history and this fact is patent in the case of projective geometry. The idea of projection of a line upon a plane is very old. It is involved in the treatment of the inversion of certain surfaces, due to Archytas (400 B.C.). Similarly the invariant property of the anharmonic or the cross ratio (AB,CD) of four colinear points A, B, C and D [it may be defined as (AC/CB)/(AD/DB), i.e. as the ratio of ratios into which C and D devide the segment AB] was essentialy recognised both by Menelaus (100 A.D.) and by Pappus (300 A.D.).

One of the first importent steps to be taken in modern times, in the development of projective geometry, was due to Desargues (1593-1662) an engineer, architect and one of the French army officers. He set forth the foundation of the theory of four harmonic points and also treated the theory of poles and polars, although not using these terms. His famous two-triangle theorem can be stated as: If two triangles, in the same plane or not, are so situated that the lines joining pairs of corresponding vertices are concurrent, then the points of intersections of corresponding [illegible] are collinear, and conversely.

In the folloing years Pascal (1623-1662) published results conics in which he gave the famous <u>mystic hexagon</u> theorem of projective geometry: If a hexagon be inscribed in a conic, then the points of intersections of the three pairs of oposite sides are collinear, and conversely.

Later in the early 19th century L.N.M. Carnot (1753-1823) published two noticable works; Geometrie de Position (1803) and the essai sur les transversales (1806). This officer in French army introducing negatives into syntheyic geometry, and exploiting the invariance of the cross ratio of four collinearpoints in which a transversal cuts a pencil of four straight lines, derived many of the classical theorems of elementary projective geometry, including those associated with complete quadrangle and complete quadrileterals.

Although Desargues and Carnot had initiated the study of projective geometry, its true independent development was launched by J.V. Poncelet (1788-1867). As a Russian prisnor of war, taken during Napoleon's retreat from Moscow, and with no book in hand, Poncelet planned his great Traite des Propriets Projective des figures. In this work he made prominent for the first time the power of central projection in demonstration and the power of the principle of continuity in research. He set forth the theory of figuers homologiques the perspective axis and the perspective center (called by Chasles the axis and center of homology) an extension of Carnot's theory of transversals, and the "Cordes ideals" of conics. Poncelet also considered the circular points at infinity and completed the first great principle of modern geometry, the principle of continuity.

The term pole, in the sense used in projective geometry, was first introduced by the french mathematician F.J.Servois (1767-1847) and the corresponding term polar by Gergonne three years later. The ideas of pole and polar was later elaborated by Gergonne and Poncelet into a regular method, out of which grew the elegent principle of duality of projective geometry.

Many of Poncelet's idea in projective geometry were further developed by the Swiss geometer J. Steiner (1796-1863). Projective geometry was finaly completely free of any metrical basis by Karl Georg Chistain von Staudt (1798-1867) in his Geometrie der Lage of 1847.

It is in Carnot's Geometrie de Position that sensed magnitudes were first systematically employed in synthetic geometry. This idea of sensed magnitudes was further exploited by A.F. Möbius (1790-1868). In his paper Der Barycentrische Calcul (1827) he used this concept to introduce homogeneous coordinates, a highly original work of all time. After this the analytical projective geometry made spectacular gains.

An interesting development in coordinate systems was inagurated by the Prussian geometer J. Plücker (1801-1868) in 1829, when he chose the straight line as the fundamental element. A point now, instead of having coordinates, posseses a linear equation. This furnished the basis of Plücker's analytical proof of the principle of duality of projective geometry. A curve may be regarded either as the locous of its points or as the envelope of its tangents.

Equipped with these two tools, the synthetic and analytical methods, projective geometry has developed into a complete self sufficiant subject. In this present work different types of structures of tetrahedra are studied using both these methods.

A pair of mutually inscribed and interlocked tetrahedra (called Möbius tetrahedra) first introduced by Möbius in 1829, was studied extensively in Mandan(5).A pair of tetrahedra in four fold perspective (called as desmic pair) was introduced by C. Stephanis in Bulletin des Sciences Mathematiques et Astronomiqus, 3(1879), 424-456. A lot of results regarding desmic tetrahedra are found in Court(2). This book also contains properties of pole and polar w.r.t. a tetrahedron. Court himself contributed a lot to the geometry of tetrahedra. A pair of perspective tetrahedra where center and plane of perspectivity are pole and polar of each other w.r.t. each of the tetrahedra are called copolar. A particular case of this is studied in Hammeed & Konnuly (1). Many other basic concepts of projective geometry can be found in Todd, Maxwell, Baker and Veblen & Young.