CHAPTER I

INTRODUCTION

1.1 The problem and its nature

The problem under consideration is to study the effects of an external magnetic field on the propagation of disturbances in an electrically conducting elastic medium. This amounts to solving the equations of motion of an elastic continuum along with Maxwell's equations governing the electromagnetic field, under appropriate boundary conditions. Suitable terms representing the interaction of the fields are to be taken into account while formulating the field equations and constitutive equations of both the fields.

The study of magneto-elastic interactions is of very recent origin compared to that of thermo-elastic interactions which date back to Duhamel (1837) and Neumann (1885), though only recently the latter has been reformulated on sound thermodynamical basis by Biot (1956) and others. The former had its beginning in Cagniard's (1952) suggestion that the observed change in the velocities of seismic waves in going from the mantle to the core of the earth, as well as the fact that only one mode seems capable of traversing the core, could be explained not on the basis of widely different elastic properties for the mantle and the core, as in the current theory, but on the basis of magneto-elastic interactions in the presence of earth's interior magnetic field. Theoretical investigation into the subject was first instituted by Knopoff (1955) and later by Baños (1956) and Chadwick (1957).

In all the works mentioned above, the interaction between the mechanical and electromagnetic fields is assumed to be completely represented by Lorentz's force in the equation of motion and a term in Ohm's law, corresponding to the electric field induced by the velocity of the material particle across the magnetic field. Further, the constitutive equation giving the response of the material to one field is supposed to remain unaffected by the presence of the other~~s~~. Though such a simplified model has not yielded significant results on a geophysical scale, the experiments recently conducted by Alen and Fleury (1963) indicate that the theoretical results may be of some significance on a laboratory scale.

1.2 Basic equations and boundary conditions

In the case of an isotropic, homogeneous, electrically conducting elastic medium, the electromagnetic field is governed by the simplified Maxwell's equations

$$\operatorname{div} \vec{B} = 0, \tag{1.2.1}$$

$$\operatorname{curl} \vec{e} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.2.2}$$

$$\operatorname{curl} \vec{H} = \vec{j}, \tag{1.2.3}$$

where $\vec{B}$ is the magnetic induction or flux density, $\vec{e}$ the electric field intensity, $\vec{H}$ the magnetic field intensity and $\vec{j}$ the current density. The simplification has been accomplished by neglecting the displacement current in writing down equation (1.2.3). The justification of this lies in the fact that frequencies associated with mechanical vibrations of or waves in an elastic conductor ~~is~~ are much smaller than those associated with electromagnetic waves of the same wave-length and that, as a result, charges are neutralized faster than they accumulate. In fact, in a mechanical conductor the charge equalization takes place in a time which is roughly 6 x 10^{-17} sec. The equation connecting the electric displacement and the charge density does not, therefore come into the picture.

To the field equations (1.2.1) to (1.2.3) are to be added the constitutive equations

$$\vec{B} = k\vec{H}, \tag{1.2.4}$$

$$\vec{j} = \sigma\left(\vec{e} + \frac{\partial \vec{U}}{\partial t} \times \vec{B}\right), \tag{1.2.5}$$

where κ is the magnetic permeability and σ the electrical conductivity of the medium, $\vec{v}$ the vector displacement of a material particle and $\vec{H}$ includes both the primary magnetic field and that induced by the motion of the particle. The vector product in the modified Ohm's law given in equation (1.2.5) denotes the electric field induced by the velocity of the material across the magnetic field.

Assuming that the elastic medium is initially unstressed and neglecting the initial stress caused by the presence of the primary magnetic field, the linearized equations governing the propagation of small disturbances, in the absence of any body force, which take into account the interaction of the mechanical and electromagnetic fields can be written in the form

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.2.6}$$

$$\tau_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij}, \tag{1.2.7}$$

$$\rho \frac{\partial^2 \vec{v}}{\partial t^2} = \mathrm{div}\, \tau + \vec{j} \times \vec{B}, \tag{1.2.8}$$

where e_{ij} stands for the strain tensor, u_i the displacement components, τ_{ij} the stress tensor, λ and μ the Lamé constants and ρ the density of the medium.

If the electric and magnetic fields in vacuum or free space are denoted by $\vec{e}$ and $\vec{h}$, the field equations to be satisfied are

$$\nabla^2 \vec{h} - \frac{1}{c^2}\frac{\partial^2 \vec{h}}{\partial t^2} = 0, \quad \nabla^2 \vec{e} - \frac{1}{c^2}\frac{\partial^2 \vec{e}}{\partial t^2} = 0, \qquad (1.2.9)$$

$$\operatorname{curl} \vec{h} = \varepsilon_0 \frac{\partial \vec{e}}{\partial t}, \quad \operatorname{curl} \vec{e} = -\kappa_0 \frac{\partial \vec{h}}{\partial t}, \qquad (1.2.10)$$

where $c = (\varepsilon_0 \kappa_0)^{-1/2}$ denotes the velocity of light and ε_0, κ_0 respectively denote the electric and magnetic permeabilities in free space.

The conditions to be satisfied at the free boundaries separating a perfectly conducting material medium and the surrounding free space are (Dunkin and Eringen, 1963)

$$[\![\vec{e} + \frac{\partial \vec{u}}{\partial t} \times \vec{B}]\!]_t = 0, \quad [\![\vec{B}]\!]_n = 0, \qquad (1.2.11)$$

$$(\tau_{ij} + M_{ij})\nu_j = \bar{M}_{ij}\nu_j, \qquad (1.2.12)$$

with

$$M_{ij} = \kappa\left(H_i H_j - \frac{1}{2} H_\alpha H_\alpha \delta_{ij}\right). \qquad (1.2.13)$$

A double bracket with a suffix t in the above equations indicates the difference in the tangential components of the quantity included and that with a suffix n implies the difference in the normal components. M_{ij} and $\bar{M}_{ij}$ denote the Maxwellian stress tensor for the material and the vacuum respectively and ν_j is the unit normal vector at a point of the boundary. If the electromagnetic radiation into the adjoining free space is to be neglected the right hand side of equation (1.2.12) will be taken equal to zero. It is to be noted that in calculating the contribution from free space to the expression on the left hand side of the first equation in (1.2.11), $\partial\vec{U}/\partial t$ is taken to be the same as that of the boundary at every point.

It should be remarked that a complete linearization of all the above equations implies that only terms which are linear in the displacement and the induced electric and magnetic fields $\vec{e}$ and $\vec{h}$ are to be retained. In particular, if $\vec{H}$ is the uniform primary magnetic field in the medium and $\vec{h}$ the induced field, making use of equations (1.2.3), (1.2.4), (1.2.6) and (1.2.7) in equation (1.2.8), the linearized equation of momentum is obtained in the form

$$\rho\frac{\partial^2\vec{U}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\vec{U}) + \mu\nabla^2\vec{U} + \kappa(\vec{H}\cdot\nabla)\vec{h} - \kappa\nabla(\vec{H}\cdot\vec{h}). \quad (1.2.14)$$

Equation (1.2.13) yields

$$M_{ij} = \kappa(H_i h_j + H_j h_i - H_\alpha h_\alpha \delta_{ij}). \quad (1.2.15)$$

Further, it may be noted that by using equations (1.2.2), (1.2.4) and (1.2.5), the induced electric and magnetic field vectors in a perfectly conducting medium can be expressed explicitly in terms of the displacement vector of a particle in the form

$$\vec{e} = -\kappa\left(\frac{\partial \vec{U}}{\partial t} \times \vec{H}\right), \qquad (1.2.16)$$

$$\vec{h} = \mathrm{curl}\,(\vec{U} \times \vec{H}). \qquad (1.2.17)$$

In deriving equation (1.2.17) $\vec{H}$ is assumed to be independent of time.

Finally, all the physical quantities involved are taken in the rationalized MKS units.

1.3 A decade of research

With a view to assess the influence of geomagnetic field on the propagation of seismic waves, Knopoff (1955) considered at length the propagation of plane elastic waves in (1) an electrically conducting infinite solid permeated by a uniform, static magnetic field, (2) an infinite solid with semi-infinite magnetic field, (3) an infinite solid with the field confined to a slab of finite thickness and (4) a semi-infinite conductor with a uniform magnetic flux. He applied the theoretical results to seismic motion in the conducting core of the earth and found that compressional waves are virtually unattenuated in the core of the earth. As mentioned in Section 1.1, Baños (1956) and Chadwick (1957) independently investigated and obtained some of the results reported by Knopoff.

Within a decade after a study was initiated purely for geophysical reasons, the development has been very rapid and most of the works are of mathematical interest. The vastness of literature on the subject dissuades us from making any attempt at an exhaustive survey. However, an attempt has been made to list as many publications as are readily available, in the Bibliography. In the following paragraphs we briefly mention some of the works. Others will be referred to in the succeeding chapters as need arises.

Kaliski and his co-workers have contributed much to the existing literature on magneto-elastics. To mention only a few, Kaliski and Petykiewicz (1960) have formulated the equations of motion for the elastic and inelastic, anisotropic conducting solids with anisotropic magnetic and electric properties. Kaliski and Rogula (1960,1961) have investigated the propagation of Rayleigh waves in a conducting elastic semi-space and on cylindrical surfaces of a conducting medium embedded in a magnetic field. Kaliski (1961) has considered the

Cauchy problem for (1) a real isotropic elastic conductor, (2) a perfect isotropic and transversely isotropic conductor and (3) an elastic dielectric in a magnetic field. Kaliski and Michalec (1963) have studied the resonance amplification of a magnetoelastic wave radiated from a cylindrical cavity in a perfectly conducting elastic medium.

Again, Kaliski has devoted many of his papers to a study of Čerenkov radiation in elastic conductors under the action of magnetic fields. More recently, in a series of papers he has derived a new set of wave equations of thermo-electro-magnetoelasticity by introducing the notion of thermal inertia in heat conduction processes. In another series of papers Nowacki has studied the plane problem of magneto-thermo-elasticity.

In order to find the possible anisotropic effects produced on the wave motion in a homogeneous, isotropic elastic medium by an applied magnetic field, Buchwald and Davis (1960) have studied the wave propagation due to an isolated body force acting at the origin in a fixed direction and varying harmonically with time. They have concluded that the anisotropic effects are negligible for small magnetic fields.

Paria (1964) has considered the effects of a uniform magnetic field on the distribution of temperature and stresses in an infinite solid having a uniform initial temperature, subjected to instantaneous plane heat sources, by using transform methods. There appears to be an error in his analysis for small time approximation.

Iya Abubakar (1964) has obtained expressions for the surface displacement for various types of sources buried in a perfectly conducting half-space with a uniform magnetic flux, assuming the resulting motion to be of SH type. Sinha (1965) has solved the problem of torsional disturbances in a conducting elastic cylinder in a magnetic field having radial and azimuthal components.

Finally, it may be mentioned that as the investigations in the following pages are entirely based upon the basic equations given in Section 1.2, except for the last chapter wherein thermal field is considered in addition to the imposed magnetic field, no work on piezoelectric, galvanomagnetic and other allied effects has been referred to.

1.4 Limitations of the linearized theory

The theory of magneto-elastic coupling given in Section 1.2 that has been widely used in the existing literature on field interactions suffers from the serious drawback of not being rigorously derived from basic conservation laws and thermodynamic principles. This means that its limitations have not yet been clearly understood.

It may be that if the strength of the applied magnetic field is large, non-linear effects become significant.

Alen and Fleury (1963) conducted experiments to study the effects of magnetic fields in the range of 10 to 20 kOe on the velocity of sound in single crystals of Cu, Ag, Al, Ta and V. They found that the quantitative agreement with the theoretical predictions, namely, the increase of the velocity as the square of the applied field and its dependence on the angle between the propagation direction and applied field, was excellent in the case of high conductivity metals, but a slight disagreement was reported at lower conductivities. Further, investigations of the effects at $4.2^{\circ}K$ in extremely high purity copper showed that for the field in certain crystallographic directions the velocity of sound no longer varied as the square of the field but increased linearly with it. Thus the linearized theory may not be applicable for metals at very low temperatures. It may be observed that unlike the results in Hydromagnetics, experimental verifications of the theoretical results in magnetoelasticity is wanting.

1.5 Scope of the present work

The following three chapters deal with the propagation of harmonic waves in a perfetly conducting elastic flat plate, with its boundaries free of mechanical tractions. In the succeeding chapter the propagation of transverse surface waves in a perfectly conducting semi-infinite medium has been considered. The last chapter is devoted to a study of the interactions of mechanical, thermal and electromagnetic fields in a conducting medium of infinite extent.

In chapter two, the problem of propagation of waves parallel to the plane boundaries of the plate is formulated for an arbitrary orientation of the magnetic field. The presence of the field is seen to result, in general, in the coupling of all the three types of waves which, following Jeffreys (1952), will be referred to in future, as P, SV and SH waves. A particular case, namely, the propagation of coupled P and SV waves in a transverse magnetic field has been studied. It is found that the electromagnetic radiation into the adjoining free space can be neglected for long and short waves as well. The dispersion equations both for symmetric and anti-symmetric modes will then be very similar to those in the non-magnetic case.

In continuation, the case of the magnetic field being parallel to the plane of polarization of the coupled P and SV waves has been investigated in chapter three. In contrast to the previous case, the electromagnetic radiation into free space cannot be neglected.

When the field is aligned with the direction of propagation, the maximum influence appears to be on long flexural waves. If the field is normal to the plate, the effect is very much pronounced on long plate waves and the flexural waves propagate almost with Alfvén velocity. When the field is oblique to the waves, the modes cannot be separated into symmetric and antisymmetric ones. This point is borne out well in the next chapter wherein the propagation of uncoupled shear waves of SH type under a field normal to the displacements of material particles has been considered at some length.

It has been shown in chapter five that uncoupled transverse surface waves which are of SH type can be propagated without any dispersion in a perfectly conducting semi-infinite elastic medium provided a uniform magnetic field acts non-aligned to the direction of wave propagation. Though the depth of penetration is found to be too large, the field interaction accounts for a slight increase in the velocity of propagation of plane shear waves in the medium.

The problem considered in chapter six is an extension of a part of Knopoff's work. It is to assess the influence of a magnetic field on thermo-elastic plane waves in an infinite medium, for different orientations of the field.