CHAPTER I

HYDRODYNAMIC AND HYDROMAGNETIC FLOW AND HEAT TRANSFER

1.1. Introduction

Towards the end of the nineteenth century, the science of fluid mechanics began to develop in two directions which had practically no point in common. On the one side there was the science of theoretical hydrodynamics which was evolved from Euler's equations of motion for a frictionless, non-viscous fluid and which achieved a high degree of completeness. However, since the results of this classical science of hydrodynamics stood in glaring contradiction to experimental results - in particular as regards the very important problem of pressure losses in pipes and channels, as well as with regard to the drag of a body which moves through a mass of fluid - it had little practical importance. For this reason, practical engineers, prompted by the need to solve the important problems arising from the rapid progress in technology, developed their own highly empirical science of hydraulics. The science of hydraulics was based on a large number of experimental data and differed greatly in its methods and in its objects from the science of theoretical hydrodynamics.

In the study of the flow properties of a fluid, we must specify the relationship between the stress-tensor and other properties of the flow, which are determinable with the help of the equations of motion. In this specification one has to rely either on some mathematical models about the structure of the fluid capable of predicting something definite about this relationship or experiments which investigate how a fluid behaves when subjected to externally imposed stresses. In the literature both of these approaches have been adopted with considerable success.

In recent years, much work was done on the generalisation of well know viscous flow solutions to take account of the additional effects of a magnetic field when the fluid is electrically conducting. When an electrically conducting fluid moves in the presence of a magnetic field, electric currents induced in the fluid modify the field and produce a mechanical force (Lorentz force) which modifies the motion. The development of the entire subject of Magnetohydrodynamics (often referred to as MHD) is based on this interaction between a moving electrically conducting fluid and an externally applied magnetic field. This new branch of fluid mechanics has not only provided a fascinating field of academic interest but, it also offers a great scope in the development of design divices in various engineering fields.

It is well known that a magnetic field imparts to the fluid a certain degree of rigidity which precludes the growth of any small random motion of the fluid. Also a magnetic field exerts a strong stabilizing influence by preventing the motion of the fluid particles across the lines of force. In particular, the inhibition of the onset of thermal convection of a layer of fluid heated from below in the presence of a magnetic field as studied by Chandrasekhar (1961) clearly explains the stabilizing influence of a magnetic field, and this inhibition, incidentally, helps to explain the phenomenon of sunspots (vide, Cowling, 1957). Furthermore, the reduction of the velocity gradient in the presence of the electromotive forces should influence turbulence and the experiments of Murgatroyd (1955) clearly reveal the dominating influence of a magnetic field in the suppression of turbulence even at the highest possible Reynolds numbers of the order of 10^5.

The study of MHD has also assumed importance in the field of aeronautics, especially missile aerodynamics, since the temperature that occur in such flight speeds are sufficient to dissociate or even ionize the air appreciably. In such high speed flights Joule heating, i.e. heating due to the flow of electric current, plays a very important role. For example, when a high speed missile (specially in the intercontinental class) re-enters the earth's atmosphere, a very large amount of heat is generated due to the friction of the air molecules and this viscous heating may sometimes be so considerable as to ionize the air near the forward stagnation point. Again for the flow past a body at sufficiently large Mach numbers, the leading edge shock wave (bow shock wave) lies close to the surface of the body and the temperature rise due to compression along the shock in such hypersonic flows may be sufficiently high to ionize the air. Since the ionized air in this stagnation region is electrically conducting, a magnetic field may be applied to it so as to induce electromotive forces in the air which in turn will be retarded. As a result, the velocity gradient decreases near the wall, implying a reduction in the skin friction. If, as is usually the case, a reduction in the skin friction implies a reduction in heat transfer (Reynolds analogy) in the ordinary hydrodynamic case, decrease in the skin friction magnetohydrodynamically should also have a similar favourable effect on the heat transfer. However, this analogy is not to be pushed very far in MHD for, in this case Joule heating may be significantly considerable to offset such thermo-mechanical analogy.

It is also well known, that the heat transfer considerations are often of crucial importance in modern engineering design. Equipment size in power production and chemical processing may be determined primarily by the attainable heat-transfer rates. A considerable fraction of the cost of many devices - for example, air-conditioning and refrigeration systems - is due to heat exchangers. In many types of equipment a successful design is possible only if provision is made to maintain reasonable temperatures by adequate heat transfer. Among such modern devices are rocket nozzles, compact electronic components, high-speed aircraft, and atmosphere-re-entry vehicles.

These many and diverse demands upon the store of knowledge concerning heat-transfer characteristics have led to a long period of sustained study of such processes. From the pioneer work of Fourier to the present, many kinds of processes have been studied both experimentally and analytically by scientists and by engineers. Many of the broad divisions of the field have been established, and satisfactory and productive theories have been presented for many types of heat transfer. There are, however, many processes of both theoretical and practical importance which are not adequately understood. In addition, new technologies give importance to types of processes not previously studied. As a result, work continues at an ever-accelerating pace to the present day.

The study of heat transfer includes the physical processes whereby thermal energy is transferred as a result of a difference, or gradient, of temperature. The information generally desired is the way in which the rate of heat transfer depends upon the various features of the process.

The three distinct modes of heat transmission are, conduction, convection and radiation. Conduction is the process in which heat is transferred from regions of higher temperature to regions of lower temperature within a system or between two systems which are in contact physically without any relative motion of the different parts of the system or systems. In fact, energy is conducted through a material in which a temperature gradient exists by the thermal motion of various of the microscopic particles of which the material is composed; energy is diffused through the material by these thermal motions. Radiation is an energy transport from material into surrounding space by electromagnetic waves. Radiant emission is also due to the thermal motion of microscopic particles, but the energy is transmitted electromagnetically. A heat (or mass) transfer process whose rate is directly influenced by fluid motion is called a convection process. The heat may be finally transferred through the flowing material by conduction, but the conduction process is basically altered by relative motion of the macroscopic particles in the fluid. Thermal energy and mass may be convected about the flow region by the motion of the fluid. For example, the disturbances connected with turbulence may have a large effect upon the transfer rate.

It should be emphasized that in most of the situations occurring in nature, heat flows by more than one of these mechanisms acting simultaneously. Their relative importance, while acting together in a heat transfer phenomenon, differs greatly with temperature. The phenomena of conduction and convection are affected primarily by temperature difference, and very little by temperature levels whereas radiation-interchange increases rapidly with increase in the temperature levels. At low temperatures, conduction and convection are the major contributors to the total heat transfer whereas at very high temperatures, radiation is the controlling factor. Recent developments in hypersonic flight, missile re-entry, rocket combustion chambers, power plants for inter-planetary flight and gas-cooled nuclear reactors have focussed attention on the thermal radiation and emphasized the need for an improved understanding of radiative transfer in these processes. In a system of fluid motion heat is transferred by conduction as well as by convection. This is quite evident from the method of heat transfer from a surface to the surrounding fluid. At the surface, heat is first transferred by conduction to the adjacent fluid elements which in turn move to regions of lower temperature (thus transferring heat by convection) and impart heat to the neighbouring fluid particles by conduction as well. In fact, it is virtually impossible to observe pure heat conduction in a fluid because, as soon as a temperature difference is imposed on a fluid, natural convection currents ensue due to the rasulting density differences. Thus convective process dominates a heat transfer phenomenon in fluid mechanics. It is, however, not possible to separate the problem of heat transfer from that of the motion of the fluid, and for this reason, a study of the hydrodynamic behaviour of the fluid is necessary in order to have an understanding of the heat transfer phenomena taking place within a moving fluid.

Convection heat transfer is classified, according to the modes of motivating flow, into forced and free convections. If the flow field is imposed by some external agency, such as a pressure gradient or a pump or blower the process is called forced convection. In this case the velocity field is independent of the temperature field whereas the temperature field is dependent on the velocity field. Mathematically, the problem reduces to finding the temperature field due to heated or cooled boundaries in a given velocity field. On the other hand when the mixing motion takes place merely as a result of the body forces on the fluid, that is, forces which are proportional to the mass or density of the fluid, the process is called free or natural convection. These body forces are generated by the density gradients resulting from heat or mass transfer, as for a domestic heating radiator or for a sun-heated road surface.

Since in free convection, temperature difference is the cause of the fluid motion which in turn changes the rate of heat transfer, the temperature and velocity fields are interdependent here. This coupling between heat transfer and fluid motion causes natural convection process to be more difficult to analyse than similar forced convection arrangements. Because, here the momentum equation and the energy equation become simultaneous equations for velocity and temperature fields. If both the convective processes are equally important in a flow system then it is called combined forced and free convection process.

In magnetohydrodynamic heat transfer problems, the additional body force term, viz. the Lorentz force comes into play in the momentum equation and the term corresponding to Joule heating appears in the energy equation. In a forced convection system, the energy equation remains uncoupled from Maxwell's equations and the Navier-Stokes equations. Thus, the electromagnetic and velocity fields can be determined independently of the temperature field. However, when natural convection forces are present the Navier-Stokes equations become coupled with the energy conservation equation, and simultaneous solution is required. If a magnetohydrodynamic device (e.g. a MHD power generator, or an electromagnetic pump) is to be intelligently designed, information should be available concerning the effects of the interactions of the electromagnetic, velocity and temperature fields.

One interesting distinction between the free and forced convection boundary layer problems in hydromagnetics may be noted. In forced convection flows, induced electromagnetic forces may modify the inviscid free stream which in turn may change the external pressure gradient or the free stream velocity which is imposed on the boundary layer. Thus for a complete solution in this case, one should also solve the inviscid free stream problem in determining the boundary layer characteristics. On the other hand in free convection problems, the velocity being zero in the free stream, the induced magnetic forces do not exist there. Thus the influence of the magnetic field on the boundary layer is exerted through the magnetic forces confined to the boundary layer only, with no additional effects arising out of the free stream pressure gradient. Thus the free convection MHD problems can be formulated in a much simpler way than the corresponding forced convection problems.

1.2. Review of the relevant literature

(a) Channel flows

(i) Hydrodynamic channel flows

Up till now numerous authors have devoted their study to flows in channels and pipes with various wall conditions relating to the hydrodynamic situations. The forced and free convection heat transfer phenomena in such flows have also received considerable attention. Here we enlist some of the works available in the existing literature mainly related to this thesis.

The behaviour of steady viscous flows around bodies depends strongly upon the streamwise pressure gradient and the normal velocity at the walls. The theory of this flow has been given independently by Jeffery (1915) and Hamel (1916). Goldstein (1938) and Rosenhead (1963) have also discussed the same problem. Pohlhausen (1921) has given closed form solution to the boundary layer equations for inflow between converging channels. Riley (1963) and Sastri (1963) have discussed the thermal boundary layers for the problem studied by Pohlhausen. The equations for slow flow through convergent or divergent channels when there is suction at one wall and blowing at the other wall have been solved by Terrill (1965a). Choudhury and Sinha (1966) and Singh and Choudhury (1967) have considered the boundary layer flow in the fully developed region and inlet length respectively, of a convergent channel with porous walls. Van Ingen (1964,1967) has considered the boundary layer flow with suction or blowing at the non-parallel walls by phase-plane representation method. He has shown that boundary layer solution can be obtained for any amount of suction or blowing at the wall in the case of inflow (convergent walls) while for outflow (divergent walls) boundary layer is possible only for sufficient amount of suction.

In recent years, a number of problems have been studied concerning the fluid flow through porous ducts, for example, in transpiration cooling and gaseous diffusion. Also, the problem of alleviating the rate of heat transfer to high speed aircraft with mass transfer or ablation cooling has been studied by several authors. Berman (1953), Sellars (1955) and Yuan (1956) have studied viscous flow in two-dimensional porous channels. Terrill (1964) has obtained more accurate results to Berman's problem for large positive Reynolds number. Terrill (1965b, 1966) has also considered flow in channel with large injection and flow through a porous annulus. Several cases of laminar flow through parallel and uniformly porous walls of different permeability have been discussed by Terrill and Shrestha (1965a, 1966, 1968). Horton and Yuan (1964-1965) have studied the laminar flow in the entrance region of a porous channel.

Tyagi (1966) has discussed the laminar forced convection of a dissipative fluid in a channel. Terrill (1965c) has studied forced convection heat transfer between two porous plates. An extensive semi-empirical study has been carried on by Elenbaas (1942a, 1942b) towards natural convection heat transfer between two parallel plates and in tubes of different cross-sections. Mull and Reiher (1930) made an experimental study of free convection heat transfer in enclosed spaces formed by two vertical walls. Batchelor

(1954) and Poots (1958) investigated the same problem theoretically and confirmed the experimental results of Mull and Reiher. Ostrach (1952) has presented an analytical study of the laminar fully developed natural convection flow of viscous fluids with and without heat sources between two vertical plates. He has also considered laminar free-convection flow about a flat plate in 1953 and combined natural and forced convection flow and heat transfer in vertical channels with linearly varying wall temperatures in 1954. Hallman (1956), Han (1959) and Lu (1959) have investigated heat transfer characteristics in free convection flow through vertical tubes with heat generation, rectangular channels and in vertical pipes with prescribed wall temperatures respectively. Tao (1960a, 1960b) has studied combined free and forced convection flow in channels, and circular and sector tubes.

(ii) Hydromagnetic channel flows

In recent years, considerable interest has developed in MHD channel flow because of its application in energy conversion schemes, e.g. power generators, electromagnetic pumps and flow meters. Hartmann (1937) considered the two-dimensional, steady Poiseuille flow of mercury between two parallel walls in the presence of an applied crossed magnetic field. Shercliff (1953, 1956) and Tanazawa (1960) have obtained exact solutions for the MHD flow in channels with rectangular and circular cross sections respectively. Gupta (1965) has discussed unsteady pipe flow at small Hartmann number. The problem of steady flow of an electrically conducting viscous fluid through uniformly porous parallel plates in the presence of transverse magnetic field has been investigated by Suryaprakasrao (1962) and Terrill and Shrestha (1964, 1965b). Terrill (1965d) has also considered the same problem when the suction velocity at one wall differs slightly from the injection velocity at the other wall. Hunt (1965) and Sloan and Smith (1966) have studied MHD flow in rectangular ducts with conducting plates. Hwang, Li and Fan (1966) have studied the laminar MHD flow in the entrance region of straight channels. Recently Sloan (1967) has considered MHD flow in an insulated rectangular duct with an oblique transverse magnetic field. Hunt and Williams (1968) have discussed the flow between parallel conducting plates when the fluid is driven by the current produced by electrodes placed one at each plane, the applied magnetic field being perpendicular to the planes. The porosity of the walls of the straight channels with transverse magnetic field has been considered by Shrestha (1967) and Reddy and Jain (1967). Jungclaus (1960) studied the hydromagnetic boundary layer flow of a viscous electrically conducting fluid between converging and diverging channels under the influence of transverse magnetic fields. Ranger (1967) has extended the Jeffery-

Hamel flow in a convergent channel to the magnetic case where the applied magnetic field is cross-radial and the plane walls are non-conducting.

A considerable amount of work in forced, natural and combined natural and forced convection MHD flow in channels has appeared in the literature. Siegel (1958), Perlmutter and Siegel (1961), and Zimin (1961) have studied the magnetic effects on the heat transfer in a fully developed flow between non-conducting parallel plates. Alpher (1961), Yen (1962) and Snyder (1964) have considered wall conductances in such flows. Globe (1959) and Shohet (1962) have considered forced convection flow in annular channels. Gupta (1960) has investigated the flow of a conducting fluid under pressure gradient between two porous parallel plates subjected to transverse magnetic field and in particular, solved the energy equation including the effects of viscous and Joulean dissipations. Nigam and Singh (1960), Shohet, Osterle and Young (1962) and Srinivasan (1963) have explored the effect of transverse magnetic field on the thermal entry length for flow between two parallel plates.

Smirnov (1959) gave an analysis of natural convection MHD flow in round vertical tubes with non-conducting walls. Gershuni and Zhukhovitski (1958) discussed free convection shear flow between vertical plates. Poots (1961) has reconsidered their problem by taking account of possible heat sources including those due to viscous and Joulean dissipations. A similar case of convective flow between vertical plates, under short circuit conditions, has been considered by Osterle and Young (1961). Young and Huang (1964) have investigated the general circuit case in which the vertical channel walls are connected through electrodes to an external circuit containing a finite load resistance R and an e.m.f. source V_{app}. Cramer (1962a, 1962b) has considered MHD free convection flows in channels and pipes and Singh and Cowling (1963) have treated free convection in a rectangular box. Srinivasan and Sastri (1964) have studied convective MHD flow between parallel porous walls. Combined natural and forced convection flow with transverse magnetic field has been studied by Mori (1961) for parallel plate channel and by Regirer (1959, 1962) for parallel plates and tubes. Singer (1965) has discussed unsteady combined convective flow through vertical channels under transverse magnetic fields. Unsteadiness is caused by prescribed variations in the axial pressure gradient and the wall temperature. The flow is assumed to be fully developed and the walls perfectly conducting. He (1966) has also considered steady, combined thermal convective MHD flow in vertical channels with finitely conducting walls whose temperatures vary along their lengths. In both the cases Singer neglects viscous and Joulean dissipations of heat.

(b) MHD boundary layer flows past a semi-infinite flat plate

Much work has been done in recent years on the generalization of the well known Blasius flow past a semi-infinite plate to include additional effects of an externally applied magnetic field, parallel to the plate. A derivation of the corresponding boundary layer equations has been carried out by Greenspan and Carrier (1959) and Davies (1963a). In the former, the fundamental equations have been replaced by Oseen equations; the linearization has been effected by assuming that the convective velocity and magnetic fields may be replaced by their free stream values. It has been found that the boundary layer thickness continues to increase with S (the square of the ratio of the Alfvén velocity to the free stream velocity), until it reaches the critical value $S = 1$, where the entire flow is brought to rest. Further discussions on the problem by means of the Oseen linearization have been given by Carrier and Greenspan (1960) and Greenspan (1960) respectively for unsteady flow conditions and the flow past a finite plate with $S > 1$. Glauert (1961) has pointed out that the linearized method of analysis, adopted by Greenspan and Carrier cannot be relied upon in problems of such complexity. Alternatively, he has obtained two solutions in series, valid for large and small values of the magnetic Prandtl number P_m (the ratio of the viscous to magnetic diffusitities). Glauert's treatment is, however, limited to the range $S < 1$ with $1-S$ not small. He has further shown that boundary layer equations cease to hold when $1-S = O(\text{Reynolds number})^{-1}$. Wilson (1964) and Stewartson and Wilson (1964), on the other hand, have shown that the solutions obtained by Greenspan and Carrier (1959) are not unique when $P_m < 1$ and $S < 1$. Goldsworthy (1961) and Reuter and Stewartson (1961) have disagreed with the interpretation by Greenspan and Carrier (1959) that the flow is 'plugged' or brought to rest at the critical value $S=1$; on the contrary they have proved that for $S=1$ the problem has no unique solution. Because, it has been improperly posed for $S=1$. Other contributions towards this problem include those due to Jungclaus (1959) and Meksyn (1962). Recently, Johnson (1966) has shown that for each S in the interval $0 < S < 1$ the problem has unique solution whereas, if $S \geqslant 1$, it has no solution. Further he has obtained that as $S \to 0$, this solution approaches the solution of the Blasius problem. Fillo (1966) has discussed the heat transfer in the problem studied by Greenspan and Carrier (1959) with viscous dissipation and Joule heating neglected.

With a view to have a better understanding of the MHD flow past a semi-infinite plate, Davies (1963a) has reconsidered the problem. He has derived the boundary layer equations, employing the usual order-of-magnitude analysis. These boundary layer equations have been solved by an iteration method suggested by Weyl (1941) for the corresponding non-magnetic case. It has been found that the drag coefficient diminishes steadily as S increases.

In the above studies the boundary layer flow was considered in the absence of pressure gradient (i.e. uniform conditions in the free stream). The influence of the external magnetodynamic pressure gradient proportional to some power of distance along the boundary has been analysed at length by Davies (1963b) and Gribben (1965a).

The MHD flow past a semi-infinite plate in the presence of a transverse or crossed magnetic field has been studied by Rossow (1958), Clauser (1963), Dix (1963), Sears (1966) and Hector (1967). Clauser concludes that the boundary layer and wake phenomena which occur in ordinary fluid mechanics do not carry over to crossed MHD flows. Both Dix and Sears have shown that there exists a Hartmann type boundary layer near the surface of the plate with a standing Alfvén wave above this Hartmann layer. Chawla (1967) has considered the fluctuating boundary layer on a semi-infinite, magnetized plate. Mittal (1968) has considered the effect of plate temperature oscillations on the MHD thermal boundary layer on a semi-infinite flat plate. Lewis (1968) has considered the boundary layer flow due to the uniform motion of a semi-infinite flat plate through an incompressible conducting fluid at rest, subject to a constant transverse magnetic field.

(c) Hydromagnetic flow due to a rotating disk

An interesting class of problems that has been studied in recent years is the rotationally symmetric hydromagnetic flows which are of great importance in MHD generators. The problem of MHD flow due to a finite rotating disk has been studied by Stewartson (1957). He has considered the effect of a vertical magnetic field on the motion of a shallow disk of mercury, part of whose base is rotating while the rest is fixed. It has been found that both in the absence of magnetic field and when the field is large there is a steady solution in which the mercury is either uniformly rotating or at rest, there being a thin layer separating the two regions. In obtaining the solution Stewartson neglects the radial component of the magnetic field in comparison with the strong external field imposed along the axis of rotation. In a note in the above problem, Majumdar (1958) has proved that for a rotationally symmetric steady hydromagnetic field the radial component of magnetic field can always be neglected in comparison with the field along the axis of rotation if the depth of the liquid column is small compared with the radius of the disk. Ray (1960) has discussed the influence of a uniform magnetic field on the couple resisting the rotation of a circular disk in a viscous conducting fluid. Sharma (1963) has examined the MHD flow over an enclosed rotating disk and found that the moment on the rotating disk increases with an increase in the Hartmann number.

The MHD flow over a rotating disk of infinite radius has been investigated by Sparrow and Cess (1962) and Kakutani (1962) with the induced magnetic fields neglected and by Sychev (1960), Rizvi (1963) and Datta (1966) with full magnetic fields. Jagadeesan (1964) has studied the hydromagnetic stagnation point flow against a rotating disk with the help of Kármán-Pohlhausen method. He has found that the torque on the disk increases as the strength of the applied magnetic field increases. Pande (1964) and Suryaprakasrao and Gupta (1966) have discussed the MHD flow against a rotating disk with large suction.

S.Datta (1964), Kelly (1964) and N.Datta (1964) have studied the effect of a uniform magnetic field on the torsional oscillations of an infinite plate in a viscous conducting fluid. Subba Raju (1966) has considered the additional effect of porosity of the disk. On the other hand, Gupta (1968) has studied the transverse waves induced by a disk oscillating about a state of steady rotation in an electrically conducting fluid with a transverse magnetic field. Contrary to the non-magnetic case (see Benney, 1965) he has found that in the presence of a magnetic field, none of the Ekman-Stokes layers becomes infinite when the angular velocity of rotation of the fluid approaches the half the frequency of oscillation of the disk.

The axisymmetric stagnation point flow in the presence of an axial magnetic field has been studied by Kakutani (1960), Meyer (1960) and Axford (1961). Kakutani has neglected the induced magnetic field and shown that the shear stress at the wall decreases with the increasing magnetic field. Axford has obtained a suitable form for the origin of the magnetic field within the body which has the property that it vanishes in the fluid far from the surface. Gribben (1965) has given a more accurate solution to the problem studied by Meyer (1960). He (1965b) has also discussed the case of the magnetic field parallel to the surface. Tiepel (1966) has considered the unsteady stagnation point flow of a viscous, electrically conducting fluid.

1.3. Inadequacy of Newtonian fluid and development of non-Newtonian fluids

On the basis of the experiments performed in water between two parallel plates, Newton stated that the stress tensor is linearly proportional to the strain-rate tensor.

Mathematically, this law can be expressed as

$$\left.\begin{aligned} p_{ij} &= -p\delta_{ij} + 2\mu e_{ij} , \\ e_{ij} &= \tfrac{1}{2}(v_{i,j} + v_{j,i}) , \end{aligned}\right\} \qquad (1.3.1)$$

where p_{ij} is the stress tensor, e_{ij} the rate-of-strain tensor, v_i the velocity vector, δ_{ij} the Kronecker delta, p an arbitrary hydrostatic pressure, μ the material constant called the coefficient of viscosity and a comma followed by a suffix denotes covariant differentiation. A fluid characterised by the linear relationship (1.3.1) is known as a Newtonian fluid. The theory based on the constitutive equations (1.3.1) provides satisfactory explanation to the phenomena like lift, skin-friction, drag, separation, secondary flow etc. (which are observed in many mobile liquids). However, it fails to explain many interesting phenomena like Merrington effect, Weissenberg effect and Reiner effect. Merrington (1943a, b) observed that when a solution of rubber in mineral oil is forced through a straight pipe, the fluid swells on emerging from the pipe. As pointed out by Merrington, this phenomenon of swelling is due to the elastic recovery of the liquid compressed in the tube. Weissenberg (1947) found that when a high polymer solution is sheared between two coaxial cylinders (keeping the inner cylinder at rest and the outer cylinder in rotation), the liquid climbs up the inner cylinder against the action of the centrifugal force. This phenomenon of climbing of the liquid in a direction perpendicular to the plane of shearing is known as Weissenberg effect or Normal stress effect. He attributed this effect to the elastic character present in the fluid. In 1957, Reiner noticed that when air is sheared in a small gap between two coaxial disks (one rotating and the other at rest), the non-rotating disk experiences a thrust at its centre. The fluids exhibiting these effects include many other industrial products like plastics, synthetic fibres, and slurries. All fluids which do not obey the equations (1.3.1) are referred to as non-Newtonian fluids. During the last two decades, many mathematical models are developed to explain the flow behaviour of these fluids. We give below a brief description of a few of these models.

(i) Reiner-Rivlin fluids

This class of fluids is governed by the constitutive equations

$$p_{ik} = -p\delta_{ik} + 2\mu e_{ik} + 4\mu_c e_{ij} e_{jk} , \qquad (1.3.2)$$

where the additional parameter μ_c is called the cross-viscosity. The coefficients μ and μ_c are, in general, functions of the three invariants of the rate-of-strain tensor besides being the functions of the material properties. The equation (1.3.2) was proposed by both Reiner (1945) and Rivlin (1948). The introduction of the cross-viscosity term in the constitutive equation is responsible for attributing to the fluid the Weissenberg and Merrington effects.

Although the equation (1.3.2) explains most of the non-Newtonian behaviours, it gives in simple shearing flow that the normal stresses in and perpendicular to the plane of shear are equal. However, in most of the real fluids, it is found that the difference of the normal stresses is a function of shear rate. Hence there is no real fluid which is governed by the equation (1.3.2).

(ii) Rivlin-Ericksen fluids

Rivlin and Ericksen (1955) considered a visco-elastic fluid which is isotropic when stationary and for which the stress components p_{ik} at any instant are expressible as polynomials in the gradients of velocity, acceleration, second acceleration, . . . , $(n-1)$th acceleration. They showed that the stress matrix $P(=\|p_{ik}\|)$ can be expressed as a matrix polynomial in n kinematic matrices $A_1, A_2, \ldots, A_n$, defined in terms of the velocity components v_i by

$$A_1 = \| A_{ik}^{(1)} \| = \| v_{i,k} + v_{k,i} \| \qquad (1.3.3)$$

and

$$A_{r+1} = \| A_{ik}^{(r+1)} \| = \| \frac{\partial A_{ik}^{(r)}}{\partial t} + v_j A_{ik,j}^{(r)} + A_{ij}^{(r)} v_{j,k} + A_{kj}^{(r)} v_{j,i} \| \quad (r = 1, 2, \cdots, n-1). \qquad (1.3.4)$$

The coefficients in this matrix polynomial are polynomial invariants of $A_1, A_2, \cdots, A_n$.

Rivlin (1955) pointed out that if the stress matrix $P (= \| p_{ik} \|)$ can be expressed as a matrix polynomial in two kinematic matrices, say $\| A_{ik} \| = A$ and $\| B_{ik} \| = B$, then P can be expressed by a relation of the form

$$P = \| p_{ik} \| = \mu_0 I + \mu_1 A + \mu_2 B + \mu_3 A^2 + \mu_4 B^2 + \mu_5 (AB + BA) + \mu_6 (A^2 B + BA^2) + \mu_7 (AB^2 + B^2 A) + \mu_8 (A^2 B^2 + B^2 A^2), \qquad (1.3.5)$$

where I is the unit matrix and μ_q $(q = 0, 1, 2, \cdots, 8)$ are polynomials in the ten matrices A, B, A^2, B^2, A^3, B^3, AB, A^2B, AB^2 and A^2B^2. If the fluid is incompressible then μ_0 can be replaced by an arbitrary quantity $-p$.

The fluids governed by the equation (1.3.5) are called Rivlin-Ericksen fluids.

Rivlin (1956) obtained exact solutions for many viscometric flows using the constitutive equations (1.3.5). His solutions accounted for shear-dependent viscosity and normal stress effects but did not account for stress relaxation effects and therefore another type of theory was given by Noll (1958), and Coleman and Noll (1960).

(iii) Coleman and Noll's second order fluids

In 1960, Coleman and Noll proposed the constitutive equation for an incompressible second order fluid as

$$p_{ik} = -p\delta_{ik} + \mu_1 A_{(1)ik} + \mu_2 A_{(2)ik} + \mu_3 A_{(1)i\alpha} A_{(1)\alpha k} , \qquad (1.3.6)$$

where $A_{(1)ik} = v_{i,k} + v_{k,i}$,

and $A_{(2)ik} = a_{i,k} + a_{k,i} + 2v_{m,i}\, v_{m,k}$,

and μ_1, μ_2 and μ_3 are respectively the coefficients of viscosity, elastico-viscosity and cross-viscosity; a_i is the acceleration vector.

The constitutive equation (1.3.5) of Rivlin-Ericksen fluids reduces to (1.3.6) when squares and products of $A_{(2)ik}$ are neglected and the coefficients of the remaining terms are taken to be constants. Markovitz and Coleman (1964) proved that μ_2 is negative and μ_3 is positive. The solution of poly-iso-butylene in cetane behaves as a second order fluid and the material constants for this solution at various concentrations were determined experimentally by Markovitz and Brown (see Truesdell, 1964).

(iv) Oldroyds Elastico-Viscous fluids

Oldroyd (1950) proposed an invariant generalization of the one-dimensional empirical rheological equations of state so that it is universally valid and can be applied to three dimensional problems. He assumed that the rheological properties of a flowing material are independent of the position and motion of the material as a whole in space but may depend at any time 't' upon the previous rheological states through which the material has passed (i.e. the rheological history which consists of the deformation history, the stress history, the temperature history, the time-lag and the physical constants associated with the material). The invariant properties of the universally valid equations of state should therefore, be independent of (a) the frame of reference, (b) the translational and rotational motions of the material element and (c) the rheological history of the neighbouring particles. This requires that instead of choosing a frame of reference fixed in space, one should consider a curvilinear coordinate system embedded in the material and convected with it as it flows, deforms or rotates. When a convected system of reference is set up with coordinate surfaces ξ^j = constant, any material element having the coordinates ξ^j will remain the same for all times and the equations describing the behaviour of a typical material element at ξ^j at time t can be expressed as functions of ξ^1, ξ^2, ξ^3 and an earlier time $t' \ (-\alpha < t' \leq t)$. The covariant metric tensor $\gamma_{jl}(\xi, t')$ at time t' will provide complete information about the distance ds between any pair of neighbouring particles ξ^j and $\xi^j + d\xi^j$ at time t', through the equation

$$[ds(t')]^2 = \gamma_{jl}(\xi, t')\, d\xi^j d\xi^l \quad . \qquad (1.3.7)$$

The deformation history of any infinitesimal material element around ξ^j, upto the time t is specified completely by the function $\gamma_{jl}(\xi, t')$ of any previous time $t' \ (\leq t)$. The stress history $\pi_{jl}(\xi, t')$

and the temperature history $T(\xi, t')$ are related to deformation history by equations of state

$$\pi_{jl}(\xi,t') = \pi'_{jl}(\xi,t') - p(\xi,t')\,\gamma_{jl}(\xi,t'), \tag{1.3.8}$$

where $p(\xi,t')$ is an arbitrary isotropic pressure, together with an invariant integro-differential equation (or set of equations) relating the following functions of t':

$$\gamma_{jl}(\xi,t'),\ \pi_{jl}(\xi,t'),\ T(\xi,t'),\ t-t' \quad (t' \leq t) . \tag{1.3.9}$$

The successive time derivatives of the metric tensor, at any material element are related to the local first, second, third, etc. rate-of-strain tensors $\eta^{(1)}_{jl}(\xi,t'), \eta^{(2)}_{jl}(\xi,t'), \ldots$, by

$$\eta^{(N)}_{jl}(\xi,t') = \frac{1}{2}\frac{\partial^N}{\partial t'^N}\left[\gamma_{jl}(\xi,t')\right], \quad (N = 1, 2, 3, \ldots) \tag{1.3.10}$$

and the incompressiblity condition can be written as

$$\gamma^{jl}(\xi,t')\,\eta^{(1)}_{jl}(\xi,t') = 0 , \tag{1.3.11}$$

where the usual summation convention is adopted for repeated indices (see Oldroyd (1965)).

In a more usual fixed system of curvilinear coordinates x^i, the rheological equations of state of the general elastico-viscous liquid will be obtained by transforming them from a convected coordinate system ξ^j. Using corresponding Greek and Latin letters to denote the convected and fixed components of the same tensor, we can write the equations of state as

$$p_{ik}(x,t) = p'_{ik}(x,t) - p(x,t)\, g_{ik}(x) , \tag{1.3.12}$$

together with a set of six equations relating to p'_{ik} (p'_{ik} is the part of stress tensor related to the change of shape of a material element) and $e^{(1)}_{ik}$. In general, this will be an integro-differential equation, involving the operations of convected differentiation and convected integration with respect to time t' following the fluid, connecting measures of the part p'_{ik} of the stress, of rate of strain, and of temperature, each primarily associated with a past time t' and with the material that later finds itself at x^i at the current time t.

The convected derivative following a material, of any tensor $b^{\cdot\cdot k \cdots}_{i \cdots}(x,t)$ is the counterpart in a fixed system of reference, of the partial time derivative in a convected system and is defined as

$$\frac{\delta b_{i\ldots}^{\cdot\cdot k\ldots}(x,t)}{\delta t} = \frac{\partial b_{i\ldots}^{\cdot\cdot k\ldots}}{\partial t} + v^m b_{i\ldots,m}^{\cdot\cdot k\ldots} + \Sigma v^m{}_{,i} b_{m\ldots}^{\cdot\cdot k\ldots} - \Sigma' v^k{}_{,m} b_{i\ldots}^{\cdot\cdot m\ldots} , \tag{1.3.13}$$

where v^i denotes a contravariant velocity vector at x^i at time t, $\Sigma\,(\Sigma')$ denotes sum of all similar terms, one for each covariant (contravariant) suffix and a comma followed by a suffix denotes a covariant differentiation. The successive strain-rate tensors have fixed components

$$\overset{(N)}{e}_{ik}(x,t) = \frac{1}{2}\frac{\delta^N g_{ik}(x,t')}{\delta t^N} . \tag{1.3.14}$$

In particular,

$$\overset{(1)}{e}_{ik}(x,t) = \tfrac{1}{2}(v_{i,k} + v_{k,i}) , \tag{1.3.15}$$

$$\overset{(2)}{e}_{ik}(x,t) = \tfrac{1}{2}\Big\{\frac{\partial v_{k,i}}{\partial t} + \frac{\partial v_{i,k}}{\partial t} + v^m v_{k,im} + v^m v_{i,km} + v^m{}_{,i} v_{k,m} + v_{i,m} v^m{}_{,k} + 2 v^m{}_{,i} v_{m,k}\Big\} . \tag{1.3.16}$$

The concept of convective coordinate allows the addition of the tensor quantities associated with different times to be added component by component yielding similar quantities as the sum.

In order to demonstrate the complete process of formulating the rheological equation of state, Oldroyd (1950) proceeds with the structural model proposed by Fröhlich and Sack (1946) which is given by

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right) p'_{ik} = 2\eta_0 \left(1+\lambda_2 \frac{\partial}{\partial t}\right) e^{(1)}_{ik} , \qquad (1.3.17)$$

where η_0 is the coefficient of viscosity, λ_1 the stress relaxation time and $\lambda_2 (< \lambda_1)$ the rate-of-strain retardation time. The relaxation time λ_1 has the physical significance that, if the motion is suddenly stopped, the stresses will decay as $\exp(-t/\lambda_1)$; and the retardation time λ_2 has the significance that, if all stresses are removed, rates of shear will decay as $\exp(-t/\lambda_2)$. The equation (1.3.17) was earlier obtained by Jeffery (1929).

If equation (1.3.17) is to be expressed in terms of spatial coordinates satisfying the requirements of material indifference then the partial derivative $\partial/\partial t$ must be replaced by the convective derivative $\delta/\delta t$, defined by (1.3.13). Thus the two obvious generalizations of equation (1.3.17) are

$$\left(1+\lambda_1 \frac{\delta}{\delta t}\right) p'_{ik} = 2\eta_0 \left(1+\lambda_2 \frac{\delta}{\delta t}\right) e^{(1)}_{ik} \qquad (1.3.18)$$

and

$$\left(1+\lambda_1 \frac{\delta}{\delta t}\right) p'^{ik} = 2\eta_0 \left(1+\lambda_2 \frac{\delta}{\delta t}\right) e^{(1)ik} . \qquad (1.3.19)$$

Liquids characterized by equations (1.3.18) and (1.3.19) are called Oldroyd liquid A and Oldroyd liquid B respectively. Oldroyd (1950,1951) made theoretical studies in simple shear behaviour of these two liquids and found that liquid B exhibits properties nearer to known experimental behaviour of non-Newtonian fluids. When equations (1.3.18) and (1.3.19) are written after substituting for the convective derivatives in spatial description, it is found that the terms, thus obtained, are linear in stresses, bilinear in stresses and velocity gradients taken together and of second degree in velocity gradients alone. Keeping this in view and introducing more constants, Oldroyd (1958) proposed an eight constant model as (in Cartesian coordinates Ox_i)

$$p'_{ik} + \lambda_1 \frac{\mathcal{D} p'_{ik}}{\mathcal{D}t} + \mu_0 p'_{jj} e_{ik} - \mu_1 \left(p'_{ij} e_{jk} + p'_{kj} e_{ij} \right) + \nu_1 p'_{jl} e_{jl} \delta_{ik} = 2\eta_0 \left\{ e_{ik} + \lambda_2 \frac{\mathcal{D} e_{ik}}{\mathcal{D}t} - 2\mu_2 e_{ij} e_{jk} + \nu_2 e_{jl} e_{jl} \delta_{ik} \right\}, \quad (1.3.20)$$

$$\frac{\mathcal{D} b_{ik}}{\mathcal{D}t} = \frac{\partial b_{ik}}{\partial t} + v_j b_{ik,j} + \omega_{ij} b_{jk} + \omega_{kj} b_{ij}, \quad (1.3.21)$$

where $\mu_0, \mu_1, \mu_2, \nu_1$ and ν_2 are five more arbitrary scalar physical constants, each with the dimensions of time occurring at high rates of shear, b_{ik} is any covariant tensor, $\omega_{ik}\left[=\frac{1}{2}(v_{k,i}-v_{i,k})\right]$ is the vorticity tensor and $\mathcal{D}/\mathcal{D}t$ denotes the material derivative which makes allowance for the rotational as well as the translational motion of the material element. Since raising or lowering of indices does not affect the material derivative $\mathcal{D}/\mathcal{D}t$ in Cartesian coordinates (or physical components of the quantities referred to orthogonal curvilinear coordinate system), equation (1.3.20), as it stands, may be considered as valid generalization of equations (1.3.18) and (1.3.19). Equation (1.3.20) represents Oldroyd liquid A when

$$\eta_0 > 0, \ \lambda_1 = -\mu_1 > \lambda_2 = -\mu_2 \geqslant 0, \ \mu_0 = \nu_1 = \nu_2 = 0 \quad (1.3.22)$$

and Oldroyd liquid B when

$$\eta_0 > 0, \ \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geqslant 0, \ \mu_0 = \nu_1 = \nu_2 = 0 \ . \quad (1.3.23)$$

Newtonian fluids can be regarded as special case of equation (1.3.20) in which all material constants except η_0 vanish.

Oldroyd (1958) has discussed a number of flow problems, viz. the shearing flow between two rotating cylinders, flow through tubes of circular and arbitrary cross-section, shearing flow between two rotating cones and found that by suitable choice of the material constants, most of the essential features observed in non-Newtonian fluids can be explained in the following manner. They possess a shear-dependent viscosity which decreases with increasing rates of shear from a limiting value η_0 at low rates of shear to a lower limiting value η_1 at high rates of shear. They exhibit Weissenberg climbing up effect and possess a distribution of normal stresses corresponding to an extra tension along the streamlines with an isotropic state of stress along the plane perpendicular to the streamlines.

Sharma (1959a) has investigated the rotation of an infinite disk in Oldroyd's elastico-viscous fluid. Rauthan (1968a, 1968b) has considered the torsional oscillation of an infinite disc and fluctuating flow near a stagnation point. Subba Raju (1967a, 1967b) has studied the forced flow against a rotating disk and flow between two rotating disks. Mishra (1965a, 1965b, 1965c) has considered Couette flow through porous walls, shear flow past a porous flat plate and heat transfer in a circular cylinder.

1.4. Basic equations

The basic equations governing the motion of an incompressible, viscous and electrically conducting liquid in the presence of a magnetic field and heat sources or sinks are, (in rationalized MKS system of units),

the equation of continuity

$$v^i_{\ ,i} = 0 \quad , \qquad (1.4.1)$$

the modified Navier-Stokes equations

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right) = p_{ij,j} + F_i + (\vec{J}\times\vec{B})_i \,, \tag{1.4.2}$$

the energy equation

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_j T_{,j}\right) = k\,T_{,jj} + \Phi + \frac{J^2}{\sigma} + Q \,, \tag{1.4.3}$$

Maxwell's equations

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t} \,, \tag{1.4.4}$$

$$\operatorname{curl}\vec{B} = \mu_e\vec{J} \,, \tag{1.4.5}$$

$$\operatorname{div}\vec{B} = 0 \tag{1.4.6}$$

and the Ohm's law

$$\vec{J} = \sigma\left[\vec{E} + \vec{q}\times\vec{B}\right] . \tag{1.4.7}$$

Again from equations (1.4.4) to (1.4.7) we get the magnetic induction equation

$$\frac{\partial\vec{H}}{\partial t} = \eta\nabla^2\vec{H} + \operatorname{curl}(\vec{q}\times\vec{H}) \,. \tag{1.4.8}$$

The various symbols are:

$v^i = \vec{q}$ the velocity vector

F_i the non-electric force per unit volume e.g. for free convection flow it is $\rho\vec{g}$, $\vec{g}$ being the force of gravity per unit mass;

p_{ij} the stress tensor related to the rate-of-strain tensor e_{ij} by various equations for different fluids as explained in the preceding section of this chapter,

$\vec{B} = \mu_e \vec{H}$ the magnetic induction vector,

$\vec{H}$ the magnetic field,

$\vec{J}$ the electric current density,

$\vec{E}$ the electric field,

T the temperature,

σ the electrical conductivity,

k the thermal conductivity,

c_p the specific heat at constant pressure,

μ_e the magnetic permeability,

ρ the density,

$(\vec{J} \times \vec{B})$ the Lorentz force per unit volume,

Q the heat source or sink per unit volume,

$\eta = \frac{1}{\mu_e \sigma}$ the magnetic diffusivity,

$\frac{J^2}{\sigma} = \sigma[\vec{E} + \vec{q} \times \vec{B}]^2$ the Joulean heat per unit volume.

In equations (1.4.1) to (1.4.3) a comma followed by a suffix denotes covariant differentiation, Φ is the viscous dissipation per unit volume and is given by

$$\Phi = p'_{ij} e_{ij} ,$$

where p'_{ij} is the deviatoric stress tensor which is, when referred to an orthogonal Cartesian frame of reference Ox_i, given by

$$p'_{ij} = p_{ij} + p(x,t)\,\delta_{ik} ,$$

p being an arbitrary isotropic pressure. ∇^2 is the Laplacian operator and for an orthogonal Cartesian frame Ox_i, it is given by

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} .$$

In deducing the above equations, the displacement current, the free electric charges and Hall currents have been neglected and the fluid has been considered to be non-magnetic. These equations govern the motion of the fluid and render a problem fully determined with the appropriate boundary conditions.

Boundary conditions

The boundary conditions to be satisfied are the usual boundary conditions imposed on the velocity (namely, the no slip condition at the boundary for viscous fluids), and the temperature fields together with certain electromagnetic boundary conditions.

The electromagnetic properties of the fluid will change abruptly to those of the solid at the solid boundary. Across such a surface of discontinuity, the electromagnetic variables have to satisfy the following conditions

(1) The normal component of the magnetic induction $\vec{B}$ is continuous across the interface

(2) If none of the regions (fluid, solid or vacuum) adjoining the boundary is perfectly conducting, the tangential component of the magnetic field $\vec{H}$ is continuous across the interface. If, however, at least one of two media in contact is perfectly conducting then the magnetic field $\vec{H}$ must satisfy the condition

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s ,$$

where $\vec{n}$ is the unit vector normal to the surface, $\vec{H}_1, \vec{H}_2$ are the values of the magnetic field on two sides of the interface and $\vec{J}_s$ is the surface current density.

(3) The tangential component of the electric field is continuous across the interface.

1.5. Outline of the thesis

This thesis deals with a few problems of fully developed laminar flow. Most of them belong to the class of Magnetohydrodynamics, the science of the motion of electrically conducting fluids under magnetic fields. However, in view of a variety of non-Newtonian fluids that are being produced by modern industries and which are, in general, electrically non-conducting, but possess some peculiar properties foreign to the ordinary viscous fluids, a chapter has been devoted to the study of an elastico-viscous fluid flowing through a porous channel. In most engineering branches, a knowledge of the temperature field in a flow problem becomes essential along with the velocity field. Further, suction and injection through the boundaries often play an important role in modifying the heat transfer and the shearing stresses at the boundaries. To make a composite study of all such phenomena we have discussed temperature fields and the influence of suction at the walls in some of the problems.

In Chapter II, a discussion has been presented in two parts, of the boundary layer flow in convergent channels. The first part deals with the MHD velocity and thermal boundary layers in a convergent channel with porous walls. The second part is a study of the thermal boundary layer of a viscous fluid containing temperature dependent heat sources and flowing between converging walls.

In Chapter III, we have considered the combined natural and free convection MHD flow in vertical channels under the influence of transversely applied magnetic field and external circuit wall conditions. The walls of the channel are having the same temperature gradient along their lengths, but they are kept at different temperatures. Consideration has been given to the heat sources present in the fluid and viscous and Joulean dissipations.

The Chapter IV has been devoted to the study of unsteady hydromagnetic boundary layer flow along a semi-infinite flat plate. Far away from the plate, the applied magnetic field as well as the free stream are parallel to the plate itself. The two-dimensional boundary layer equations have been separated into two sets of equations, one representing the steady part and the other the unsteady part of the motion. These two sets of equations are solved in sequence.

In Chapter V, we have discussed the unsteady hydromagnetic forced flow against a rotating disk of infinite radius. A suitable set of similarity transformations has been introduced and the system of reduced equations has been solved by Kármán-Pohlhausen method.

Chapter VI is devoted to a theoretical investigation of the problem of the flow and heat transfer of an elastico-viscous fluid in a porous channel. The constitutive equations used are those proposed by Oldroyd (1958). The temperatures of the walls are varying linearly along their lengths in the major flow direction. The velocity and temperature profiles are obtained for various values of the suction and visco-elastic parameters and the corresponding viscous drags and the Nusselt number at the walls are computed.