CHAPTER - I

POSSIBILITY OF A POLYTROPIC RELATION IN HOMOLOGOUS RADIAL MOTION OF A GRAVITATING GAS SPHERE WITH EQUATION OF STATE INVOLVING TEMPERATURE AND RADIATIVE HEAT TRANSFER. *

1. Introduction - As stated in the general introduction, the question of the possibility of a polytropic pressure density relation for special types of motion of a gravitating gas sphere has been raised in this chapter. The gas mass has a continuous generation of heat from subatomic energy sources. To simplify the discussion of the problem we have brought in the additional characteristic of homology, defined by the eqn.(7) of Art.2. Homology means that contraction or expansion takes place under a law of similarity. Further, opacity and subatomic energy generation are taken as products of powers of density and temperature. The modified question stands thus : "Under what conditions the radial motion of a gravitating gas sphere with subatomic energy generation and heat transfer through radiation satisfies the further conditions of polytropy and homology, if

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* We are indebted to Prof. M.H.Martin for some correspondence and obtaining some comments on it from Prof. J.M.Burgers regarding the presentation. New matter has since been added by us.

the opacity and energy generation obey power laws?". It may be noted in this connection that homology and some power laws of opacity and energy generation are very often assumed in astrophysical literature. The object of our investigation is the study of a gas sphere under special physical conditions just stated. The gas configuration may in certain cases extend upto infinity. Some conclusions from the study, as we shall see later, may be applicable to actual stellar bodies. The way of handling the equations has evolved through the work of previous authors and is detailed in Art.3 after the basic equations have been set up.

2. Basic equations - The equation of momentum of a gravitating gas sphere is

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2}, \tag{1}$$

where m and t are taken as independent variables, so that $\partial/\partial t$ means differentiation following the motion and r is the radius of the sphere enclosing mass m, G is the gravitational constant and P, the pressure.

The equation of continuity is

$$4\pi r^2 \rho \frac{\partial r}{\partial m} = 1 , \tag{2}$$

ρ being the density.

The equation of energy gives

$$C_V \frac{\partial T}{\partial t} + P \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = -\frac{\partial}{\partial m}\left(-\frac{16\pi^2 ac}{3k} r^4 \cdot \frac{\partial T^4}{\partial m}\right) + \epsilon \tag{3}$$

In this equation, following Thomas [8]; no notice has been taken of radiative and material viscosity. The left hand side of this equation represents the rate of gain of heat per unit mass and is equated to the right hand side, the first term of which represents the net flux of energy and the second term ϵ represents the subatomic energy generation per unit mass. Here C_V is treated as a constant. The quantities a and c in the bracket of the right hand side of (3) are the Stefan-Boltzmann constant and the velocity of light respectively. Furthermore, the energy generation ϵ and opacity k are assumed to obey power laws given by

$$\epsilon = C\rho^{\alpha}T^{\beta} \quad ; \quad k = K\rho^{\mu}T^{\lambda} \tag{4}$$

where C, α, β, K, μ and λ are constants.

The equation of state is

$$P = R\rho T. \tag{5}$$

In this equation the radiation pressure has been neglected in comparison with gas pressure. On the right hand side of (3) the radiational flux has been taken into account. This only means that the result will hold in such ranges of temperature, density etc. that whereas radiation pressure and radiation energy are negligible in comparison with the gas pressure and energy, the flux of radiation (or to be more accurate the derivative of the flux multiplied by r^4/k) is not negligible when multiplied by ac. This assumption is necessary for successful application of our technique but it turns out that for certain cases discussed, n turns out to be 4/3. In all such cases the assumption is not necessary as will be shown in Art.5 where the case $n = 4/3$ occurs for the first time. Though the sub-atomic energy generation takes place at the cost of mass but following the general practice the loss of mass, which indeed is very small (as in Proton-Proton interaction or Carbon Nitrogen cycle), has been neglected.

When (4) is substituted in (3) we have the four eqns.(1), (2), (3) and (5) to determine the four unknown quantities P, ρ, r and T. We now wish to investigate the conditions under which a polytropic relation of the form

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^n \tag{6}$$

holds, considerations being restricted to homologous motion i.e., motions given by

$$r = r_0(m) \cdot f(t) \quad ; \quad f(0) = 1 . \tag{7}$$

This means that special solutions of the determinate set of equations (1), (2), (3), (5) are sought having additional restrictions (6) and (7). The general solutions of (1), (2), (3), (5) are difficult to obtain. It will be found that solutions with special conditions (6) and (7) exist.

The relations between P, ρ, T and their initial values are given by

$$\rho = \frac{\rho_0(m)}{f^3} \; ; \quad P = \frac{P_0(m)}{f^{3n}} \; ; \quad T = \frac{T_0(m)}{f^{3n-3}} , \tag{8}$$

using (2), (6) and (5) respectively. Substituting (8) in (1) and (3), we get

$$r_0 \cdot \frac{d^2 f}{dt^2} + 4\pi r_0^2 \cdot f^{2-3n} \cdot \frac{dP_0}{dm} + \frac{Gm}{r_0^2 f^2} = 0 \qquad (9)$$

$$3\,\frac{df}{dt} \cdot f^{2-3n} \cdot (c_p - n c_v)\, T_0 = f^{-\lambda'} \cdot \frac{d}{dm}\left(\frac{16\pi^2 ac}{3K_0}\, r_0^4 \,\frac{dT_0^4}{dm}\right) + \epsilon_0 f^{-\mu'} \qquad (10)$$

respectively, where

$$\left.\begin{aligned} &\epsilon_0 = C \rho_0^{\alpha} T_0^{\beta} \quad ; \quad K_0 = K \rho_0^{\mu} T_0^{\lambda} \\ &\lambda' = 12n - 16 - 3\mu - \lambda(3n-3) \; ; \; \mu' = 3\beta(n-1) + 3\alpha . \end{aligned}\right\} \qquad (11)$$

To handle the eqns.(9) and (10), we use the lemma due to Bandyopadhyay [1].

Lemma. - If $\phi_i(m)$ and $f_i(t)$ $(i = 1, 2, 3)$ are all continuous and differentiable functions and neither of them vanishes and if

$$\phi_1(m) \cdot f_1(t) = \phi_2(m) \cdot f_2(t) + \phi_3(m) \cdot f_3(t)$$

then either f's are mutually proportional and ϕ's are connected by a suitable linear relation or vice versa. In the light of this lemma, eqn.(9) implies

either (I) $$\frac{d^2 f}{dt^2} = \frac{A_1}{f^{3n-2}} = \frac{B_1}{f^2} \qquad (12)$$

and

$$-r_0 = \frac{4\pi r_0^2}{A_1}\,\frac{dP_0}{dm} + \frac{Gm}{B_1 r_0^2} \qquad (13)$$

or (II) $$-r_0 = \frac{4\pi r_0^2}{A_2}\,\frac{dP_0}{dm} = \frac{Gm}{B_2 r_0^2} \qquad (14)$$

and

$$\frac{d^2 f}{dt^2} = \frac{A_2}{f^{3n-2}} + \frac{B_2}{f^2} \tag{15}$$

similarly eqn.(10) implies

either (III)

$$f^{2-3n} \cdot \frac{df}{dt} = \frac{A_3}{f^{\lambda'}} = \frac{B_3}{f^{\mu'}} \tag{16}$$

and

$$3T_0 \left(C_p - nC_v\right) = \frac{1}{A_3} \cdot \frac{d}{dm}\left(\frac{16\pi^2 ac}{3K_0} r_0^4 \frac{dT_0^4}{dm}\right) + \frac{\epsilon_0}{B_3} \tag{17}$$

or (IV)

$$3T_0 \left(C_p - nC_v\right) = \frac{1}{A_4} \cdot \frac{d}{dm}\left(\frac{16\pi^2 ac}{3K_0} r_0^4 \frac{dT_0^4}{dm}\right) = \frac{\epsilon_0}{B_4} \tag{18}$$

and

$$\frac{df}{dt} \cdot f^{2-3n} = \frac{A_4}{f^{\lambda'}} + \frac{B_4}{f^{\mu'}} \tag{19}$$

provided $C_p - nC_v \neq 0$ and the constants A's and B's occurring in the above equations are all non-Zero.

3. Review of previous work and its relation with the present work - The basic equations (1) - (5) are well known. L.H.Thomas* [8] investigated the conditions under which a homologous contraction of a star can take place with acceleration neglected. His chief interest was in stellar evolution and has therefore obtained the conditions under which

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* Thomas considered energy generation in stability problem (M.N.R.A.S., 91, 619(1930). Our problem, however, is entirely different from that of Thomas.

equations (1) with $\partial^2 r/\partial t^2 = 0$, (2), (3) with $\epsilon = 0$, (4) and (5) can have a solution of the form (7). Using (1), (2), (7) and (5) he was led to the same relations as one would get on putting $n = 4/3$ in (8). Substitution of these in (3) leads to (10) with $n = 4/3$ and the second term on the right hand side dropped. n turned out to be 4/3 in the case of Thomas because from the equilibrium equation (with $\partial^2 r/\partial t^2 = 0$) P was found to be $P_0(m)/r^4$. Separation of this equation was then evident but would not have been possible if the energy generation term was present. Bandyopadhyay [1] took up the same problem as Thomas [8] and could take into consideration energy generation term through the lemma just referred to. Pal and Bandyopadhyay [6] investigated equations (1), (2), (5), (6) and (7). Relations in (8) were deduced from (2) and (7) and then substituted in (1). This lead to an equation like (9).

We have combined these two methods which lead to eqns. (9) and (10). Investigation of the previous authors and the work in the present thesis can be schematically epresented as follows (equations below refer to the equations of his chapter, sometimes modified as stated) :

	Thomas	Bandyopadhyay	Pal and Bandyopadhyay	Present thesis
ter- ning ua- ons.	Equilibrium eqn. i.e. Eqn.(1) with $\ddot{r}=0$ Eqn. of continuity(2) Eqn. of energy i.e. Eqn. (3) with $\epsilon=0$ Eqn. of state (5)	Equilibrium eqn. i.e. Eqn.(1) with $\ddot{r}=0$ Eqn. of continuity(2) Eqn. of energy with $\epsilon\neq 0$ Eqn. of state (5)	Eqn. of motion (1) with $\ddot{r}\neq 0$ Eqn. of continuity(2) Adiabatic eqn. i.e. Eqn.(6) with $n=\gamma$	Eqn. of motion (1) with $\ddot{r}\neq 0$ Eqn. of continuity (2) Eqn. of energy(3) with $\epsilon\neq 0$ Eqn. of state(5)
peci- l ypes of olution ought.	$r = r_0(m)\cdot f(t)$	$r = r_0(m)\cdot f(t)$	$r = r_0(m)\cdot f(t)$	$r = r_0(m)\cdot f(t)$ $\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^n$
imple ela- ions bta- ned.	$P = \frac{P_0(m)}{f^4}$ $\rho = \frac{\rho_0(m)}{f^3}$ $T = \frac{T_0(m)}{f}$	Same as those of Thomas	$P = \frac{P_0(m)}{f^{3\gamma}}$; $\gamma = \frac{c_p}{c_v}$ $\rho = \frac{\rho_0(m)}{f^3}$ $T = \frac{T_0(m)}{f^{3\gamma-3}}$	$P = \frac{P_0(m)}{f^{3n}}$ $\rho = \frac{\rho_0(m)}{f^3}$ $T = \frac{T_0(m)}{f^{3n-3}}$
uation or uations - Separation	Equivalent to Energy equation (10) with $\epsilon=0$	Equation of energy(10) with $\epsilon\neq 0$	Equation of motion (9).	Equation of motion (9) Equation of energy (10) with $\epsilon\neq 0$.

The combination of equations in the case of previous authors was rather simple. In the present thesis the manner in which the equations have been combined produces more complications arising out of mutually interlocked possibilities. The results are therefore more manifold though only a few of them have direct bearing with known opacity and energy generation laws of Astrophysics.

The lemma just stated is very natural to occur to anyone who might come across with an equation of the type (9) or (10). As far as we know, however, this was first obtained by Bandyopadhyay [1]. This was also proved by him (unpublished) without assuming differentiability. Martin [5] has independently used a similar lemma with four terms and later obtained it in a more general form. Birkhoff [4] has also used the lemma independently. The interest of the method centers essentially round the question of separation.

Art.4 <u>Method of handling equations (9) and (10)</u> - In the next four sections we have considered the various possibilities that arise out of the four combinations (I) and (III), (I) and (IV), (II) and (III), (II) and (IV). Each case gives rise to a set of equations in $f(t)$ and a set of equations which together with (2) and (5) determine the initial configuration. One or both of them may or may not be determinate. Our procedure in all the sections runs in the following order. We first deal with the equations to determine $f(t)$ getting certain restrictions on α, β, λ and μ. We next examine the equations determining the initial configuration. In some cases these also give rise to further restrictions on α, β etc., which together with the previously obtained relations now give the restrictions on α, β, λ and μ in the final form. Sometimes the restrictions on initial distribution breaks up into different sub-cases giving different alternative restrictions on α, β etc. Each sub-case is taken separately along with restrictions imposed on α, β etc., by equations connecting f's.

In the following sections we will drop the subscript zero determining the initial configuration as no ambiguity is caused thereby.

5. Combinations (I) and (III) - Equations (12) and (16) to determine $f(t)$ are overdetermined. In (12), second term equated to the third gives $A_1 = B_1$ and $3n-2=2$ i.e., $n = 4/3$ while (16) similarly implies $A_3 = B_3$ and $\lambda' = \mu'$. Calculating d^2f/dt^2 from (16) and identifying it with (12), we get $\lambda' = 5/2$ and $A_1 = -A_3^2/2$. Equations (13) and (17) along with (2), (4) and (5) ascertain the initial configuration. It can be seen that these equations are invariant with regard to a scale transformation so that any instant can be taken as $t=0$. These equations are determinate in the sense that the number of unknowns equals the number of equations and thus these do not give any restriction on α, β etc. Equation (11) with $\lambda' = \mu' = 5/2$ and $n = 4/3$ (as obtained above) gives

$$3\alpha + \beta = \frac{5}{2} \quad ; \quad 3\mu + \lambda = -\frac{5}{2} \qquad (20.1)$$

giving the final restrictions on α, β, λ and μ. This shows that ϵ and k must be of the form

$$\epsilon = C\left(\frac{\rho}{T^3}\right)^{\alpha} T^{\frac{5}{2}} \quad ; \quad k = K\left(\frac{\rho}{T^3}\right)^{\mu} T^{-\frac{5}{2}} \tag{20.2}$$

where α and μ may take up any value.

The form of $f(t)$ is determined by integrating (16) with $n = 4/3$ and $\lambda' = 5/2$ as

$$f = \left(1 + \tfrac{3}{2} A_3 t\right)^{\frac{2}{3}} \tag{21}$$

This shows that $f \longrightarrow \infty$ (i.e., the gas sphere explodes) as $t \longrightarrow \infty$ or $f = 0$ in a finite time $t_0 = -2/3A_3$ according as $A_3 \gtrless 0$. $f = 0$, however, is not physically attainable as the gas becomes degenerate after a finite time.

In this case, it may be shown that our above conclusions will still hold even if we include terms due to radiation pressure and radiation energy in (5) and (3) respectively. This is shown as follows :

Eqn.(5) will be replaced by

$$p = R\rho T + \tfrac{1}{3} a T^4 \tag{5.1}$$

while (3) will be

$$\frac{\partial}{\partial t}\left(c_V T + \frac{1}{\rho}\cdot a T^4\right) + \left(R\rho T + \tfrac{1}{3} a T^4\right)\cdot\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)$$
$$= -\frac{\partial}{\partial m}\left(-\frac{16\pi^2 ac}{3k} r^4 \frac{\partial T^4}{\partial m}\right) + \epsilon \tag{3.1}$$

Substitution of $\rho = \rho_0(m)/f^3$ (which being a consequence of (2) and (7) still holds in this case) in (6) with $n = 4/3$ gives $P = P_0(m)/f^4$. With these values of P and ρ, eqn.(5.1) becomes

$$P_0 = R\rho_0 \cdot Tf + \frac{1}{3} a (Tf)^4 .$$

which gives on differentiation with respect to t,

$$\frac{\partial}{\partial t}(Tf) \cdot \left[R\rho_0 + \frac{4}{3} a (Tf)^3 \right] = 0$$

This implies that $\partial/\partial t \cdot (Tf) = 0$ (since the quantity in the squared bracket is positive), so that Tf is a function of m only and hence $T = T_0(m)/f$. Now with $\rho = \rho_0(m)/f^3$ and $T = T_0($ the term $\partial/\partial t\,(aT^4/\rho) + aT^4/3 \cdot \partial/\partial t\,(\frac{1}{\rho})$ of (3.1) arising out of radiation vanishes identically and thus establishes our proposi

6. <u>Combinations (I) and (IV)</u> - Eqns. (12) and (19) to determine $f(t)$ are overdetermined. Eqn.(12) gives $n = 4/3$ an $A_1 = B_1$ as in Art.5. Calculating d^2f/dt^2 from (19) and equating to (12), we get

$$-\frac{A_1}{f^2} + \frac{(2-\lambda')A_4^2}{f^{2\lambda'-3}} + \frac{(2-\mu')B_4^2}{f^{2\mu'-3}} + \frac{(4-\lambda'-\mu')A_4 B_4}{f^{\lambda'+\mu'-3}} = 0 \qquad (22)$$

This equation will be an identity if any one of the following conditions holds:-

(a) All the terms contain the same power of f with the sum of their coefficients vanishing.

(b) Three terms on the left hand side of (22) contain the same power of f with the sum of their coefficients vanishi

While the coefficient of the fourth term vanishes.

(c) Two terms contain the same power of f with the sum of their coefficients vanishing while the coefficients of the other two terms vanish separately.

(d) Terms paired two by two have same power of f with the sum of their Coefficients vanishing.

(e) Coefficient of each term vanishes separately.

The case (a) gives $\lambda' = \mu' = 5/2$. For case (b), since $A_1 \neq 0$, the three terms which contain the same powers of f must involve the first term and any two of the rest. This leads to the same conclusion as (a). The case (c) leads to either of the following possibilities (since A's and B's are all nonzero)

(I) $\lambda' = \frac{5}{2}$; $2 - \mu' = 0$; $4 - \lambda' - \mu' = 0$

(ii) $\mu' = \frac{5}{2}$; $2 - \lambda' = 0$; $4 - \lambda' - \mu' = 0$

(iii) $\lambda' = 2$; $\mu' = 2$; $\lambda' + \mu' = 5$

These three cases are to be ruled out as inconsistent. The case (d) gives the possibilities

(i) $2 = 2\lambda' - 3$; $2\mu' - 3 = \lambda' + \mu' - 3$

(ii) $2 = 2\mu' - 3$; $2\lambda' - 3 = \lambda' + \mu' - 3$

(iii) $2 = \lambda' + \mu' - 3$; $2\lambda' - 3 = 2\mu' - 3$

All these lead to the same conclusion as (a) i.e., $\lambda' = \mu' = 5/2$

Hence $\lambda' = \mu' = 5/2$ together with $n = 4/3$

leads to the restrictions on α, β given by (20) of Art.5.

Eqns. (13) and (18) together with (2) and (5) giving the initial configuration are overdetermined and will presumably give further restrictions on the parameters. Eqn.(18) with the first term equated to the third gives

$$3T(c_p - nc_v) = \frac{\varepsilon}{B_4} = \frac{c\rho^\alpha T^\beta}{B_4}$$

which gives

$$\rho^\alpha T^{\beta-1} = \frac{3(c_p - nc_v)\cdot B_4}{c} \tag{23}$$

This will be true if any one of the following holds:

(i) $\rho =$ Constant ; $\beta = 1$

(ii) $T =$ Constant ; $\alpha = 0$

(iii) $\rho =$ constant ; $T =$ Constant

(iv) $\alpha = 0$; $\beta = 1$

(v) $\rho \propto T^{(1-\beta)/\alpha}$; $\alpha \neq 0$, $\beta \neq 1$.

Cases (ii) and (iii) are not considered here as they make the first term in the right hand side of (10) vanish and will be considered in Case (5) of Article 9. Case (iv) is ruled out being inconsistent with $3\alpha + \beta = 5/2$ of (20). Hence we consider the cases (i) and (v).

<u>Case (i)</u> Putting ρ = constant in (2) and integrating we get $m = 4/3 \cdot \pi r^3 \rho$ which we write as $m = \lambda_1 r^3$. Changing the variable m in (13) to r with $m = \lambda_1 r^3$ and integrating

We obtain

$$P = -\frac{3}{8}\left(1+\frac{G\lambda_1'}{B_1}\right)\cdot\frac{\lambda_1 A_1}{\pi} + \text{constant.}$$

This gives from (5) with ρ = constant

$$T = P_1 r^2 + Q_1 \tag{24}$$

where P_1 and Q_1 are constants.

Now eqn.(18) with the first term equated to the second gives after using $m = \lambda_1 r^3$ and (11),

$$3T(c_p - nc_v) = \frac{64\pi^2 ac\rho^{-\mu}}{27\lambda_1^2 K A_4}\cdot\frac{1}{r^2}\frac{d}{dr}\left(r^2 T^{3-\lambda}\frac{dT}{dr}\right)$$

Elimination of r between (24) and the above equation gives

$$P_1\left[(18-4\lambda)T^{3-\lambda} - 4(3-\lambda)Q_1 T^{2-\lambda}\right] - MT = 0 \tag{25}$$

where $M = 81 A_4 K\lambda_1^2\rho^{\mu}(c_p - nc_v)/64\pi^2 ac$ which is clearly non-zero. Again $P_1 \neq 0$ for otherwise eqn.(25) would imply $T = 0$ which is not possible. Using arguments similar to those applied to (22) and remembering that neither M nor P_1 is zero, eqn.(25) will be an identity if

$$3 - \lambda = 1 \quad ; \quad P_1(18-4\lambda) - M = Q_1 = 0$$

$Q_1 = 0$ gives from (24), $T = P_1 r^2$ and $\lambda = 2$.

The restrictions on α, β etc. in the final form are thus $\lambda = 2$ and $\lambda' = \mu' = \frac{5}{2}$. These imply from (20) with $n = 4/3$ and $\beta = 1$

$$\alpha = \frac{1}{2}, \quad \beta = 1 \quad ; \quad \mu = -\frac{3}{2}, \quad \lambda = 2$$

so that $$\epsilon = c\rho^{\frac{1}{2}}T \quad ; \quad k = K\rho^{-\frac{3}{2}}T^2 \tag{26}$$

Case (V).– Here $\rho = L\,T^{(1-\beta)/\alpha}$, L being a constant. This gives from (5),

$$P = R\,L\,T^{(1-\beta+\alpha)/\alpha} \qquad (27)$$

Now, eqn.(13) can be written as

$$-\left(r^3 + \frac{4\pi \cdot r^4}{A_1} \cdot \frac{dP}{dm} \right) = \frac{Gm}{B_1}$$

which gives on differentiation with respect to m,

$$-\left[3r^2 \frac{dr}{dm} + \frac{4\pi}{A_1} \frac{d}{dm}\left(r^4 \frac{dP}{dm} \right) \right] = \frac{G}{B_1}.$$

Using (2) and (27), the above equation can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right) + A\,T^{m'} = \mathcal{B}, \qquad (28.1)$$

where

$$A = \frac{4\pi L G A_1 \alpha}{B_1 R(\alpha-\beta+1)} \;;\; \mathcal{B} = \frac{-3A_1\alpha}{R(\alpha-\beta+1)} \;;\; m' = \frac{1-\beta}{\alpha}. \qquad (28.2)$$

Clearly A, $\mathcal{B}$ and m' are all non-zero. Using (2) and $\rho = L\,T^{(1-\beta)/\alpha}$ eqn.(18) with its first term equated to the second can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 T^{n'} \frac{dT}{dr} \right) = Q\,T^{m'+1}, \qquad (29.1)$$

where

$$n' = \frac{(\beta-1)(\mu+1)}{\alpha} - \lambda + 3 \;;\; Q = \frac{9KA_4(c_p - nc_v)L^{\mu+2}}{4ac}. \qquad (29.2)$$

Eliminating d^2T/dr^2 from (28.1) and (29.1) and writing $\Phi($
for $(dT/dr)^2$, we get

$$\Phi(T) = \frac{1}{n'}\left[A T^{m'+1} + Q T^{m'-n'+2} - BT\right] \qquad (30)$$

Further, $\Phi = (dT/dr)^2$ gives from (28.1),

$$r = \frac{2\sqrt{\Phi}}{B - AT^{m'} - \frac{1}{2}\frac{d\Phi}{dT}}.$$

Elimination of r between the above equation and $\Phi = (dT/dr$
leads to

$$2(B-AT^{m'})\frac{d\Phi}{dT} - \frac{3}{4}\left(\frac{d\Phi}{dT}\right)^2 + \Phi\left(2Am'T^{m'-1} + \frac{d^2\Phi}{dT^2}\right) = (B-AT^{m'}$$

Substitution of the values of $d\Phi/dT$ and $d^2\Phi/dT^2$ from (
in the above equation gives

$$a_1 T^{m'} + b_1 T^{m'-n'+1} + c_1 T^{2m'} + d_1 T^{2m'-n'+1} + e_1 T^{2(m'-n'+1)} + f_1 = 0 \qquad (31)$$

where

$$\left.\begin{aligned} a_1 &= AB(4n'^2+8n'-2m'^2+m'+3)/2n'^2 \;;\; b_1 = BQ(m'-n'+2)(6n'-2m'+1)/2n'^2 \\ c_1 &= A^2(m'^2-2m'-4n'^2-8n'-3)/4n'^2 \;;\; d_1 = AQ(m'^2+6n'^2-m'-11n'-m'n'-2)/2n'^2 \\ e_1 &= -Q^2(m'-n'+2)(n'+2-m')/4n'^2 \;;\; f_1 = -B^2(3+4n'^2+8n')/4n'^2 \end{aligned}\right\} \qquad (32)$$

The possibility $a_1 = b_1 = c_1 = d_1 = e_1 = f_1 = 0$ for (31) to be
an identity is ruled out for equations in (32), in view of
$A, B, Q \neq 0$, will then impose too many restrictions on m and

It has been analysed and found that they are not satisfied simultaneously. Other possibilities are ruled out from simil considerations. This case is thus altogether ruled out.

The form of $f(t)$ for the case (i) considered is given by (21) and has been already discussed in Art.5.

7. Combinations (II) and (IV) - Here the eqns. (15) and (1 to determine $f(t)$ are overdetermined. Calculating d^2f/dt^2 fr (19) and identifying it with that in (15), we get

$$\frac{A_2}{f^{3n-2}}+\frac{B_2}{f^2}-\frac{A_4^2(3n-2-\lambda')}{f^{2\lambda'+5-6n}}-\frac{B_4^2(3n-2-\mu')}{f^{2\mu'+5-6n}}-\frac{A_4B_4(6n-4-\lambda'-\mu')}{f^{\lambda'+\mu'+5-6n}}=0 \quad (33)$$

Using arguments similar to those given for (22) and remembering that none of A_2, B_2, A_4 and B_4 is zero, eqn.(33) will be an identity if any one of the following conditions ho

i) $\lambda'=\mu'=\frac{5}{2}$; $n=\frac{4}{3}$; $A_2+B_2=-(A_4+B_4)^2/2$ (34

ii) $\lambda'=-\frac{1}{2}$; $\mu'=\frac{1}{2}$; $n=\frac{2}{3}$; $A_2=+A_4^2/2$; $B_2=-B_4^2/2$. (34

iii) $\lambda'=\frac{1}{2}$, $\mu'=-\frac{1}{2}$; $n=\frac{2}{3}$; $A_2=B_4^2/2$; $B_2=-A_4^2/2$. (34

iv) $\lambda'=2$, $\mu'=\frac{3}{2}$; $n=\frac{7}{6}$; $A_2=-\frac{A_4B_4}{2}$; $B_2=-\frac{A_4^2}{2}$. (34

v) $\lambda'=\frac{3}{2}$, $\mu'=2$; $n=\frac{7}{6}$; $A_2=-\frac{A_4B_4}{2}$; $B_2=-\frac{B_4^2}{2}$. (34

vi) $\lambda'=4$, $\mu'=3$; $n=\frac{5}{3}$; $A_2=-A_4^2$; $B_2=-A_4B_4$. (34

vii) $\lambda'=3$, $\mu'=4$; $n=\frac{5}{3}$; $A_2=-B_4^2$; $B_2=-A_4B_4$ (34

Eqns. (14) and (18) together with (2) and (5) to determine the initial configuration being overdetermined, will give further restrictions on α, β etc. In (14), the first term equated to the third gives $m = \lambda_2 r^3$, $(\lambda_2 = -B_2/G)$. This means ρ = constant. Again in (14) the first term equated to second gives $dP/dm = -A_2/4\pi r$ which with $m = \lambda_2 r^3$ gives on integration

$$P = -\frac{3\lambda_2 A_2}{8\pi} r^2 + \text{Constant}.$$

Using (5), the above equation gives

$$T = -\frac{3\lambda_2 A_2}{8\pi R \rho} r^2 + Q_2 = P_2 r^2 + Q_2, \text{ say.} \tag{35}$$

This equation is similar to (24). Again in (18), the first term equated to the last implies (23) which gives rise to the same four cases as in Art.6. Here the case (i) of Art.6. viz.,

ρ = constant, $\beta = 1$ only need be considered (as we have here a homogeneous distribution only). This case as in Art.6, leads to $\lambda = 2$ along with $\beta = 1$.

Thus the net restrictions on α, β etc. in their final form are obtained by combining $\lambda = 2$, $\beta = 1$ (as obtained from mass distribution) with the eqns.(34.1) - (34.7) (as obtained from equations to determine $f(t)$) as follows :

i) $\alpha = \frac{1}{2}$, $\beta = 1$; $\lambda = 2$, $\mu^* = -\frac{3}{2}$; $n = \frac{4}{3}$

so that

$$\epsilon = C\rho^{\frac{1}{2}} T \quad ; \quad k = K\rho^{-\frac{3}{2}} T^2 \tag{36.1}$$

ii) $\alpha = \frac{1}{2}$, $\beta = 1$; $\lambda = 2$, $\mu = -\frac{11}{6}$; $n = \frac{2}{3}$

so that

$$\epsilon = C\rho^{\frac{1}{2}}T \quad ; \quad k = K\rho^{-\frac{11}{6}}T^2 \tag{36.2}$$

iii) $\alpha = \frac{1}{6}$, $\beta = 1$; $\lambda = 2$, $\mu = -\frac{13}{6}$; $n = \frac{2}{3}$

so that

$$\epsilon = C\rho^{\frac{1}{6}}T \quad ; \quad k = K\rho^{-\frac{13}{6}}T^2 \tag{36.3}$$

iv) $\alpha = \frac{1}{3}$, $\beta = 1$; $\lambda = 2$, $\mu = -\frac{5}{3}$; $n = \frac{7}{6}$

so that

$$\epsilon = C\rho^{\frac{1}{3}}T \quad ; \quad k = K\rho^{-\frac{5}{3}}T^2 \tag{36.4}$$

v) $\alpha = \frac{1}{2}$, $\beta = 1$; $\lambda = 2$, $\mu = -\frac{3}{2}$; $n = \frac{7}{6}$

so that

$$\epsilon = C\rho^{\frac{1}{2}}T \quad ; \quad k = K\rho^{-\frac{3}{2}}T^2 \tag{36.5}$$

vi) $\alpha = \frac{1}{3}$, $\beta = 1$; $\lambda = 2$, $\mu = -\frac{4}{3}$; $n = \frac{5}{3}$

so that

$$\epsilon = C\rho^{\frac{1}{3}}T \quad ; \quad k = K\rho^{-\frac{4}{3}}T^2 \tag{36.6}$$

vii) $\alpha = \frac{2}{3}$, $\beta = 1$; $\lambda = 2$, $\mu = -1$; $n = \frac{5}{3}$

so that

$$\epsilon = C\rho^{\frac{2}{3}}T \quad ; \quad k = K\rho^{-1}T^2 \tag{36.7}$$

The form of $f(t)$ in case $n = 4/3$ remains the same as in (21) but in other cases f is given by

$$\frac{df}{dt} = \frac{A_4}{f^{\lambda' - 3n+2}} + \frac{B_4}{f^{\mu' - 3n+2}} \quad ; \quad \lambda' \neq \mu'.$$

The behaviour of f will depend on the signs of A_4 and B_4. The initial motion is one of expansion if both A_4 and B_4 be positive and is one of contraction if both are negative. If A_4 and B_4 be of opposite signs the initial motion will be one of contraction or expansion according as the negative quantity amongst A_4 and B_4 is numerically less or greater than the other.

8. Combinations (II) and (III) - Here the eqns.(15) and (16) to determine f are overdetermined. In (16), the second term equated to the third gives $\lambda'=\mu'$; $A_3=B_3$. Again calculating d^2f/dt^2 from (16) and identifying it with that in (15), we get

$$\frac{A_2}{f^{3n-2}}+\frac{B_2}{f^2}-\frac{A_3^2\,(3n-2-\lambda')}{f^{2\lambda'+5-6n}}=0$$

Since none of A_2, B_2 and A_3 is zero, the above equation will be an identity if either of the following holds:

i) $3n-2=2=2\lambda'+5-6n$; $A_2+B_2-A_3^2(3n-2-\lambda')=0$

ii) $3n-2=2$; $3n-\lambda'-2=A_2+B_2=0$

Case (i) with $\lambda'=\mu'$ gives $\lambda'=\mu'=5/2$ and $n=4/3$ leading to (20) while case (ii) leads to $\lambda'=\mu'$ and $n=4/3$ giving from (16), df/dt = constant i.e., $d^2f/dt^2=0$. This case, however, will be discussed in Art.9(1).

Eqns.(14) and (17) together with (2) and (5) giving the initial configuration being overdetermined will give further restrictions on α, β etc. As in Art.7, eqn.(14) gives ρ = constant and $m=\lambda_2 r^3$ and these lead to (35). Using $m=\lambda_2 r^3$, eqn.(17) gives

$$T = M_2 \cdot \frac{1}{r^2}\frac{d}{dr}\left(r^2 T^{3-\lambda}\frac{dT}{dr}\right) + N T^{\beta} \tag{37.1}$$

where

$$M_2 = \frac{64\pi^2 ac\rho^{-\mu}}{81 A_3 (c_p - nc_v) K \lambda_2^2} \quad ; \quad N = \frac{C\rho^{\alpha}}{3 B_3 (c_p - nc_v)} \tag{37.2}$$

Clearly M_2 and $N \neq 0$. Elimination of r between (37.1) and (35) gives

$$T - \left[(18-4\lambda) M_2 P_2 T^{3-\lambda} - 4 Q_2 (3-\lambda) M_2 T^{2-\lambda} \cdot P_2 + N T^{\beta}\right] = 0$$

Using arguments similar to those given for (22), the above equation will be an identity if any one of the following conditions holds:

i) $\beta = 1$, $N = 1$, $9-2\lambda = Q_2 = 0$

ii) $\lambda = 1$, $\beta = 3-\lambda$, $2M_2P_2(9-2\lambda) + N = 0$

iii) $\beta = 2-\lambda$, $\lambda = 2$, $1 - 2M_2P_2(9-2\lambda) = -N + 4M_2P_2Q_2(3-\lambda) = 0$

Thus the restrictions on α, β etc. in their final form are obtained by combining (20) (as obtained from equations to determine $f(t)$) with each of the above four cases as follows:

i) $\alpha = \frac{1}{2}$, $\beta = 1$; $\mu = -\frac{7}{3}$, $\lambda = +\frac{9}{2}$

so that

$$\epsilon = C\rho^{\frac{1}{2}} T \quad ; \quad k = K\rho^{-\frac{7}{3}} T^{+\frac{9}{2}} \tag{38.1}$$

ii) $\alpha = \frac{1}{6}$, $\beta = 2$; $\mu = -\frac{7}{6}$, $\lambda = 1$

so that

$$\epsilon = C\rho^{\frac{1}{6}} T^2 \quad ; \quad k = K\rho^{-\frac{7}{6}} T \qquad (38.2)$$

(iii) $\alpha = \frac{5}{6}$, $\beta = 0$; $\lambda = 2$, $\mu = -\frac{3}{2}$

so that

$$\epsilon = C\rho^{\frac{5}{6}} \quad ; \quad k = K\rho^{-\frac{3}{2}} T^2 \qquad (38.3)$$

n turns out to be $4/3$ in each case.

The form of $f(t)$ here is also given by (21).

9. Zero Cases: - The cases when some of the terms in (9) or (10) vanish separately or simulteneously are now taken up. As before we drop the subscript zero.

In (9), d^2f/dt^2 and dP/dm may vanish separately in different cases but not simultaneously for $d^2f/dt^2 = dP/dm = 0$ leads to $m=0$ which is impossible.

In (10), one or more of the quantities $c_p - nc_v$, ϵ , $d/dm \cdot (16\pi^2 ac r^4/3k \cdot dT^4/dm)$ may vanish but $df/dt = 0$ is not possible for this means no motion of the gas sphere. Again $T=0$ is not possible for this means $P=0$ which is absurd. It may be noted that $\epsilon = 0$ means that energy generation is absent and consequently does not put any restrictions on ρ or T .

We will now consider the combinations of all possible cases arising out of (9) with those of (10). The case where $c_p - nc_v = 0$ and no other term vanishes is discussed separately in Art.10.

(1). The term d^2f/dt^2 only of (9) vanishes and no term of (10) vanishes - In this case, eqn.(9) implies $n = 4/3$ and

$$4\pi r^4 \frac{dP}{dm} + Gm = 0 \tag{39}$$

Eqn.(10) implies with $n = 4/3$, $\lambda' = \mu' = 2$ and

$$3H(c_p - nc_v)T = \frac{d}{dm}\left(\frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm}\right) + \epsilon \tag{40}$$

where $H = df/dt$ = constant. Eqns.(39) and (40) together with (2) and (5) determining the initial distribution are determinate and do not impose further restrictions on the parameters.

Thus the restrictions on α, β etc. in their final form are given by $\lambda' = \mu' = 2$ which becomes on using (11),

$$3\alpha + \beta = 2 \quad ; \quad 3\mu + \lambda = -2 \tag{41.1}$$

i.e.

$$\epsilon = C\left(\frac{\rho}{T^3}\right)^{\alpha} T^2 \quad ; \quad k = K\left(\frac{\rho}{T^3}\right)^{\mu} T^{-2} \tag{41.2}$$

The motion in this case is uniform since df/dt = constant.

(2). $dP/dm = 0$ in (9) - This means P = constant and (9) leads to

$$\frac{Gm}{r^3} = -K_1 \quad ; \quad \frac{d^2f}{dt^2} = \frac{K_1}{f^2} \tag{42.1}$$

K_1 being a constant. The former implies $m \propto r^3$ i.e. ρ = constant. Now P = constant and ρ = constant lead to T = constant from (5) so that (10) with $\epsilon = C\rho^{\alpha}T^{\beta}$ reduces to

$$3\frac{df}{dt} \cdot f^{2-3n+\mu'}(c_p - nc_v) = C\rho^{\alpha} T^{\beta - 1}$$

Since ρ and T are constants, this equation gives

$$\frac{df}{dt} = \text{Constant} \times f^{3n-2-\mu'} \; ; \quad \frac{d^2f}{dt^2} = \text{Constant} \times f^{6n-5-2\mu'} \qquad (42.2)$$

Comparing the two values of d^2f/dt^2 from (42.1) and (42.2), we have

$$n = \frac{2(\alpha-\beta)+1}{2(1-\beta)} \qquad (43)$$

giving the restriction on the parameters in their final form.

The distribution is homogeneous in this case and the gas sphere is isothermal and the nature of motion is given by first equation of (42.2) which has been discussed in Art.5. The motion is not vibratory in this case.

(3). <u>$d^2f/dt^2 = 0$ in (9) and $c_p - nc_v = 0$ in (10)</u> - As in Case (1), eqn.(9) with $d^2f/dt^2=0$ leads to eqn.(39) and $n = 4/3$ wh eqn.(10) with $c_p - nc_v = 0$ leads to $\lambda' = \mu'$ and

$$\frac{d}{dm}\left(\frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm}\right) + C\rho^{\alpha} T^{\beta} = 0 \qquad (44)$$

Eqns.(39) and (44) together with (2) and (5) determi the initial configuration and they are determinate.

Thus the net restrictions on the parameters are give by $\lambda' = \mu'$ with $n = 4/3$ which with (11) can be written as

$$3\alpha + \beta = -(3\mu + \lambda) = m \quad , \quad (\text{Say})$$

so that

$$\epsilon = C\left(\frac{\rho}{T^3}\right)^{\alpha} T^{m} \; ; \quad k = K\left(\frac{\rho}{T^3}\right)^{\mu} T^{-m} \qquad (45)$$

μ, m being arbitrary.

This includes as a special case $\alpha = \mu = m = 0$ which means ϵ and k are constants. Since $d^2f/dt^2 = 0$, and $c_p - nc_v =$ the equations to determine the initial configuration are the same as the equilibrium equations of a stellar model. Therefor in the light of the fact that ϵ and k are constants, initial configuration in this case, reduces to Eddington's model (also known as standard model) and due to similarity transformation remains so at all instants.

The motion is uniform in this case since $d^2f/dt^2=0$.

(4). $dp/dm=0$ in (9) and $c_p - nc_v = 0$ in (10) - The former impli P = constant and leads to form(9) the two equations in (42.1). Thus in this case also ρ = constant and T = constan so that eqn.(10) with $c_p - nc_v = 0$ gives $\epsilon = 0$.

Thus in this case the distribution is homogeneous and $\epsilon = 0$.

The nature of the motion is given by the first equation of (42.

(5). $d^2f/dt^2=0$ in (9) and $\frac{d}{dm}\left(\frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm}\right) = 0$ in (10) - Eq with $d^2f/dt^2=0$ gives (39) and $n = 4/3$. The second condition gives from (10), with df/dt = constant,

$$\mu' + 2 - 3n = 0 \qquad (46)$$

$$\rho^\alpha T^{\beta-1} = \text{Constant} \; ; \quad \frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm} = \text{Constant} = S_1 . \qquad (47)$$

Eqn.(46) with $n = 4/3$ and (11) gives

$$3\alpha + \beta = 2 . \qquad (48)$$

The initial distribution is given by eqns.(47) and (39) together with (2) and (5). These equations are overdeterminate. The first equation of (47) leads to the cases (i) - (v) of Art.6(p.14) of which case (iv) is ruled out being inconsistent with (48). Case (iii) is also ruled out for it gives $dP/dm = 0$ and this with $d^2f/dt^2 = 0$ gives $m = 0$ from (9).

Case (i) - Here ρ = constant gives from (2), $m = \lambda_1 r^3$. This gives from (39), $P = P_1 r^2 + Q_1$. Eqn.(5) then gives $T = P_2 r^2 + Q_2$. Elimination of r between $T = P_2 r^2 + Q_2$ and the second equation of (47) gives $T^{3-\lambda}(T-Q_2)^{3/2}$ = constant. This will be an identity if $\lambda = 9/2$, $Q_2 = 0$, so that $\epsilon = C(\rho/T^3)^{\alpha} T^2$; $k = K\rho^{\mu} T^{9/2}$; $3\alpha+\beta = 2$.

Case (ii) - In this case, the second equation of (47) is satisfied identically. The initial configuration is determined from (39), (2) and (5) so that in this case $\epsilon = CT^2$; $k = K\rho^{\mu} T^2$; $3\alpha+\beta = 2$.

Case (v) - Proceeding exactly in the same way as Art.6, eqn.(39) reduces, after using (2) and $\rho = L\,T^{(1-\beta)/\alpha}$ to (28.1) with $B = 0$. The second equation of (47) reduces to (29.1) with $Q = 0$. Thus this case is also ruled out as in Art.6.

(6). $\frac{d^2f}{dt^2} = 0$ in (9) and $\epsilon = 0$ in (10) - The former implies df/dt = constant = H and eqn.(9) leads to $n = 4/3$ and (39) while (10) implies

$$3n - 2 - \lambda' = 0 \tag{49}$$

$$3H(C_p - nC_v)T = \frac{d}{dm}\left(\frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm}\right) \tag{50}$$

Eqn.(49) with $n = 4/3$ gives $\lambda' = 2$ and the eqn.(50) with (39), (2) and (5) gives the initial configuration which is determinate.

Thus the restriction on the parameters in their final form is given by $\lambda' = 2$ which with (11) and $n = 4/3$ gives $3\mu+\lambda = -2$

so that

$$\epsilon = C \rho^{\alpha} T^{\beta} \quad ; \quad k = K \left(\frac{\rho}{T^3}\right)^{\mu} T^{-2} \quad .. \tag{51}$$

The motion is uniform in this case.

10. <u>Case $c_p - n c_v = 0$</u> - This case is of special importan for it leads to $n = c_p/c_v$, the true adiabatic exponent. In t case we have in place of (III) and (IV) the following equatio (after dropping the subscript zero)

(V)

$$\frac{A_3}{f^{\lambda'}} = \frac{B_3}{f^{\mu'}} \tag{52}$$

and

$$\frac{1}{A_3} \frac{d}{dm}\left(\frac{16\pi^2 ac}{3k} r^4 \frac{dT^4}{dm}\right) + \frac{\epsilon}{B_3} = 0 \tag{53}$$

Eqns.(52) and (53) may be combined with either (I) or (II), i turn.

<u>Combinations (I) and (V).</u> In this case eqns.(12) and (52) determine $f(t)$ are overdetermined and give $n = 4/3$, $A_1 =$ and $\lambda' = \mu'$, $A_3 = B_3$ respectively.

The initial configuration given by (13) and (53) together with (2) and (5) is determinate and gives no restric on the parameters.

Thus the restrictions on α, β etc. in their fina form are given by $\lambda' = \mu'$ with $n = 4/3$ and these give from (

$$3\alpha + \beta + 3\mu + \lambda = 0 \tag{54.}$$

which means

$$\epsilon = c\left(\frac{\rho}{T^3}\right)^{\alpha} T^{m} \;;\; k = K\left(\frac{\rho}{T^3}\right)^{\mu} T^{-m} \quad .. \tag{54}$$

where

$$3\alpha + \beta = -(3\mu + \lambda) = m \quad .$$

The form of $f(t)$ is obtained by integrating (12) as

$$\left(\frac{df}{dt}\right)^2 = S - \frac{2A_1}{f} \tag{55}$$

S being a constant. Eqn.(55) shows that df/dt can not vanish for two distinct values of f. Hence vibration is not possible in this case.

Eqn.(55) has explicit solution given by

$$\sqrt{S}\,t + \text{constant} = \pm\left[\frac{S}{2A_1} f\sqrt{f^2 - \frac{2A_1}{S}} + \frac{2A_1}{S}\log\left\{f + \sqrt{f^2 - \frac{2A_1}{S}}\right\}\right] ;\ (S>0, A_1>0). \tag{56}$$

$$\sqrt{S}\,t + \text{constant} = \pm\left[\frac{S}{2A_1} f\sqrt{f^2 - \frac{2A_1}{S}} - \frac{2A_1}{S}\log\left\{f + \sqrt{f^2 - \frac{2A_1}{S}}\right\}\right] ;\ (S>0, A_1<0) \tag{56}$$

$$\sqrt{-S}\,t + \text{constant} = \pm\left[\frac{2A_1}{S}\sin^{-1}\sqrt{\frac{fS}{2A_1}} - \sqrt{f\left(f - \frac{2A_1}{S}\right)}\right] ;\ (S<0, A_1<0). \tag{56}$$

Combinations (II) and (V) - Eqns. (15) and (52) to ascertain $f(t)$ are overdetermined and (52) implies $\lambda' = \mu'$, $A_3 = B_3$. Eqns.(14) and (53) together with (2) and (5) to determine the initial configuration are also overdetermined. Here also as in Art. 7, eqn. (14) gives ρ = constant and eqn. (35). Substitution of T form (35) in (53) gives.

$$P_2\,(18 - 4\lambda)\,T^{3-\lambda} - 4P_2Q_2(3-\lambda)\,T^{2-\lambda} + N_1 T^{\beta} = 0 \quad . \tag{57}$$

where $N_1 = 27\, C \lambda_1^2 K \rho^{\alpha+\mu} / 64\pi^2 ac$ which is clearly non-zero. Eqn.(57) will be an identity if any one of the following conditi holds :

(i) $\beta = 3-\lambda \quad , \quad Q_2 = 0 \quad , \quad N_1 + P_2(18-4\lambda) = 0$ (58.1)

(ii) $\beta = 2-\lambda \quad , \quad \lambda = \frac{9}{2} \quad , \quad N_1 + 6 P_2 Q_2 = 0$ (58.2)

Thus the restrictions on α, β etc. in their final for are obtained by combining $\lambda' = \mu'$ (as obtained from equations determining f) with any one of (58.1) and (58.2). Using (11), $\lambda' = \mu'$ and (58.1) imply

$$\beta + \lambda = 3 \quad ; \quad n = \frac{7 + 3(\mu+\alpha)}{3} \tag{59.1}$$

so that

$$\epsilon = C \rho^{\alpha} T^{\beta} \quad ; \quad k = K \rho^{\mu} T^{3-\beta} \tag{59.2}$$

while $\lambda' = \mu'$ and (58.2) imply

$$\beta = -\frac{5}{2} \quad ; \quad \lambda = \frac{9}{2} \quad ; \quad n = \frac{10 + 3(\mu+\alpha)}{6} \tag{60.1}$$

so that

$$\epsilon = C \rho^{\alpha} T^{-\frac{5}{2}} \quad ; \quad k = K \rho^{\mu} T^{\frac{9}{2}} \tag{60.2}$$

The form of $f(t)$ is given by (15). Eqn.(14) shows that $B_2 < 0$ (as $B_2 = -Gm/r_0^3$) and A_2 will be $>$ or < 0 according as dP_0/dm is $<$ or > 0. In a stellar distribution pressure decreases towards the surface and hence

$dP_0/dm < 0$ but $dP_0/dm > 0$ is possible in a gas. sphere. The cases are considered below.

Case.(i) $dP_0/dm < 0$ (i.e $A_2 > 0$) - Eqn.(15) in this case can be written in the form

$$-\frac{1}{C_1 f^2}\frac{d^2 f}{dt^2} = \frac{D_1}{f^{3n}} + \frac{1}{f^4} \tag{61}$$

where $C_1 = -B_2 > 0$, $D_1 = A_2/B_2 < 0$. Putting

$$f(t) = \frac{\xi(t)}{(-D_1)^{\frac{1}{4-3n}}}\ ; \quad p_1^{\,2} = \frac{C_1}{(-D_1)^{-\frac{3}{4-3n}}} \tag{62}$$

eqn.(61) becomes

$$\frac{d^2\xi}{dt^2} = p_1^{\,2}\left(-\frac{1}{\xi^2} + \frac{1}{\xi^{3n-2}}\right) \tag{63}$$

This gives on integration,

$$\left(\frac{d\xi}{dt}\right)^2 = p_1^{\,2}\left(-F + \frac{2}{\xi} - \frac{2}{3(n-1)}\cdot\frac{1}{\xi^{3n-3}}\right) \tag{64}$$

F being a constant. The character of motion determined by (64) has been investigated by Pal and Bandyopadhyay [6] based on the following consideration:

For vibration the conditions are

(1) $d\xi/dt$ and therefore the right hand side of (64) should vanish for two values of ξ, viz., $\xi_1 > 1$ and $\xi_2 < 1$

(2) The value of $d^2\xi/dt^2$ must be negative for $\xi = \xi_1$; and positive for $\xi = \xi_2$.

(3) The time taken in passing from ξ_1 to ξ_2 should be finite. This will be ensured provided the roots, ξ_1 and ξ_2

(of the right hand side of (64) equated to zero) are not repeated roots.

These considerations led Pal and Bandyopadhyay to conclude that the gas sphere will vibrate if $n > 4/3$ and the constant F appearing in (64) satisfies the inequality

$$0 < F < 2\left(1 - \frac{1}{3n-3}\right)$$

No physical interpretation of this inequality was suggested by Pal and Bandyopadhyay. A simple interpretation, however, can be given. Using (62), (64) and $f(0) = 1$, F can be expressed in terms of $(df/dt)_0$, the initial value of df/dt as follows

$$F = \frac{2(3n-3+D_1)}{3(n-1)\,(-D_1)^{\frac{1}{4-3n}}} - \left(\frac{df}{dt}\right)_0^2 \cdot (-D_1)^{\frac{1}{4-3n}}$$

Hence in our present case the above inequality becomes

$$0 < \frac{2(3n-3+D_1)}{3(n-1)} - \left(\frac{df}{dt}\right)_0^2 \cdot (-D_1)^{\frac{6}{4-3n}} < \frac{2(3n-4)}{3(n-1)} \cdot (-D_1)^{\frac{1}{4-3n}} \qquad (65)$$

Thus the above inequality on F implies that the value of the initial velocity lies within the limits given by

$$\frac{2(3n-3+D_1)}{3(n-1)} \cdot (-D_1)^{-\frac{6}{4-3n}} - \frac{2(3n-4)}{3(n-1)}(-D_1)^{-\frac{5}{4-3n}} < \left(\frac{df}{dt}\right)_0^2 < \frac{2(3n-3+D_1)}{3(n-1)} \cdot (-D_1)$$

It may be remarked that (61) has also been obtained by Bhatnagar [3] following Bhatnagar and Kothari [2]. This was obtained by them without considering heat exchange. Their investigations are related to anharmonic pulsations of stars under adiabatic conditions following the Darwin lecture by Rosseland [7]. Our present investigation shows that the type of anharmonic pulsation of a homogeneous gas sphere considered by the previous authors is possible even with heat exchange under suitable laws for opacity and energy generation.

(ii) $dP_0/dm > 0$ $(i.e\ A_2 < 0)$.- Eqn.(15) in this case can be w written as

$$-\frac{1}{C_2 f^2}\frac{d^2 f}{dt^2} = \frac{D_2}{f^{3n}} + \frac{1}{f^4} \tag{66}$$

where $B_2 = -C_2 < 0$; $D_2 = A_2 / B_2 > 0$

Putting $f(t) = \xi(t)/(D_2)^{\frac{1}{4-3n}}$; $p_2^2 = C_2/(D_3)^{-\frac{3}{4-3n}}$, eqn.(66) becomes

$$\frac{d^2\xi}{dt^2} = -p_2^2\left[\frac{1}{\xi^{3n-2}} + \frac{1}{\xi^2}\right]$$

Since $\xi > 0$, $d^2\xi/dt^2$ can never vanish for any value of ξ and hence $d\xi/dt$ can not vanish for two values of ξ . . Thus vibration is not possible in this case.

11. Conclusions. - The investigation enables us to have th question in a form more precise than what we had at the beginni and hence before writing the conclusions we restate our questic in a more precise form, "Do there exist suitable values of α, λ, μ, C_p and C_v (which are characteristics of the physical system) for which given suitable initial distribution and initial radial velocity, a motion of the required type (i.e. homologous and polytropic) is possible ? If so what are

the restrictions on the abovementioned parameters and the nature of the initial distributions and also the values of n in these cases ? " It may be noted that the initial configuration and velocity together with certain given values of α, β etc. determine the motion altogether i.e. determine n and the form of $f(t)$.

In answering the question, all logically possible cases have been considered. A model having a distribution expanding upto infinity has been included and only cases like $m \leq 0$ or $\rho =$ have been ruled out. The initial configuration has been considered as determinate when the number of variables equals the number of equations. The exact nature of the initial configuration has been examined only under very obvious circumstances (e.g., when it is homogeneous)*. The nature of $f(t)$ has been sometimes studied in details. The general nature (vibratory, expanding to a limit or to infinity) of the motion has been obtained in most of the cases and in explicit forms in some cases.

Our conclusion is that the required type of motion is possible if any one of the set of conditions holds as tabulated in next few pages.

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* Cases in which the initial configuration has not been studied are of such complication that even the corresponding equilibrium models have not yet been studied in astrophysical literature except in some very special cases by elaborate numerical methods

Cases	GIVEN				CONSEQUENCES		Remarks
	ϵ	k	Initial configurations Initial velocity $\propto r_0(m)$	$\frac{C_p}{C_v}$	n	Nature of motion	
(I) & (III)	$C\left(\frac{\rho}{T^3}\right)^{\alpha} T^{\frac{5}{2}}$	$K\left(\frac{\rho}{T^3}\right)^{\mu} T^{-\frac{5}{2}}$	Eqns. (13), (17), (2) and (5).	$\neq \frac{4}{3}$	$\frac{4}{3}$	$\frac{d^2f}{dt^2} = \frac{\text{Constant}}{f^2}$	
(I) & (IV)	$C\rho^{\frac{1}{2}} T$	$K\rho^{-\frac{3}{2}} T^2$	Homogeneous	$\neq \frac{4}{3}$	$\frac{4}{3}$	$\frac{d^2f}{dt^2} = \frac{\text{Constant}}{f^2}$	These forms of ϵ and k are special cases of the above for $\alpha = \frac{1}{2}$, $\mu = -\frac{3}{2}$
(II) & (IV)	$C\rho^{\frac{1}{2}} T$	$K\rho^{-\frac{3}{2}} T^2$	Homogeneous	$\neq \frac{4}{3}$	$\frac{4}{3}$	$\frac{df}{dt} = \frac{A_4}{f^{\lambda'-3n+2}} + \frac{B_4}{f^{\mu'-3n+2}}$	These and some other formulae do not correspond to the energy generation ϵ and opacity k of known stellar models.
	$C\rho^{\frac{1}{2}} T$	$K\rho^{-\frac{11}{6}} T^2$		$\neq \frac{2}{3}$	$\frac{2}{3}$		
	$C\rho^{\frac{1}{6}} T$	$K\rho^{-\frac{13}{6}} T^2$		$\neq \frac{2}{3}$	$\frac{2}{3}$		
	$C\rho^{\frac{1}{3}} T$	$K\rho^{-\frac{5}{3}} T^2$		$\neq \frac{7}{6}$	$\frac{7}{6}$		
	$C\rho^{\frac{1}{2}} T$	$K\rho^{-\frac{3}{2}} T^2$		$\neq \frac{7}{6}$	$\frac{7}{6}$		
	$C\rho^{\frac{1}{3}} T$	$K\rho^{-\frac{4}{3}} T^2$		$\neq \frac{5}{3}$	$\frac{5}{3}$		
	$C\rho^{\frac{2}{3}} T$	$K\rho^{-1} T^2$		$\neq \frac{5}{3}$	$\frac{5}{3}$		

Cases	Given: ϵ	Given: κ	Given: Initial configurations. Initial velocity $\propto r_0(m)$	$\frac{c_p}{c_v}$	Consequences: n	Consequences: Nature of motion	Remarks
(II) & (III)	$c\rho^{\frac{1}{2}}T$	$K\rho^{-\frac{7}{3}}T^{\frac{9}{2}}$	Homogeneous	$\neq\frac{4}{3}$	$\frac{4}{3}$	$\frac{d^2f}{dt^2}=\frac{\text{Constant}}{f^2}$	Due to homogeneity the powers of ρ in ϵ or κ appear not to matter but may occur in the physical laws for ϵ or κ. Here ρ and so ϵ and κ change with time.
	$c\rho^{\frac{1}{6}}T^2$	$K\rho^{-\frac{7}{6}}T$		$\neq\frac{4}{3}$	$\frac{4}{3}$		
	$c\rho^{\frac{5}{6}}$	$K\rho^{-\frac{3}{2}}T^2$		$\neq\frac{4}{3}$	$\frac{4}{3}$		
Art. 9(1)	$C\left(\frac{\rho}{T^3}\right)^{\alpha}T^2$	$K\left(\frac{\rho}{T^3}\right)^{\mu}T^{-2}$	Eqns. (39), (40), (2) and (5).	$\neq\frac{4}{3}$	$\frac{4}{3}$	$\frac{df}{dt}=\text{Constant}$	
Art. 9(2)	Anything	Anything	Homogeneous configuration with constant pressure.	$\neq\beta_1$	$\frac{2(\alpha-\beta)+1}{2(1-\beta)}\ (=\beta_1)$	$\frac{d^2f}{dt^2}=\frac{\text{Constant}}{f^2}$	Here no stellar boundary with $P=0$ is possible.
Art. 9(3)	$C\left(\frac{\rho}{T^3}\right)^{\alpha}T^m$	$K\left(\frac{\rho}{T^3}\right)^{\mu}T^{-m}$	Eqns. (39), (44), (2) and (5).	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{df}{dt}=\text{Constant}$	For the special case $\alpha=\mu=m=0$, the configuration at any instant is the same as that of Eddington's model.

Cases	Given: ϵ	k	Initial configurations. Initial velocity $\propto r_0(m)$	$\frac{c_p}{c_v}$	Consequences: n	Nature of motion	Remarks
Art. 9(4)	0	$K\rho^{\mu}T^{\lambda}$	Homogeneous configuration with constant pressure	Anything	$\frac{c_p}{c_v}$	$\frac{d^2f}{dt^2}=\frac{\text{Constant}}{f^2}$	Here no stellar boundary with $P=0$ is possible.
Art. 9(6)	0	$K\left(\frac{\rho}{T^3}\right)^{\mu}T^{-2}$	Eqns. (50), (39), (2) and (5)	$\neq\frac{4}{3}$	$\frac{4}{3}$	$\frac{df}{dt}=\text{Constant}$	
(I) & (V)	$C\left(\frac{\rho}{T^3}\right)^{\alpha}T^{m}$	$K\left(\frac{\rho}{T^3}\right)^{\mu}T^{-m}$	Eqns. (13), (53), (2) and (5)	$\frac{4}{3}$	$\frac{4}{3}$	$\left(\frac{df}{dt}\right)^2=S-\frac{2A_1}{f}$	
(II) & (V)	$C\rho^{\alpha}T^{\beta}$ $C\rho^{\alpha}T^{-\frac{5}{2}}$	$K\rho^{\mu}T^{3-\beta}$ $K\rho^{\mu}T^{\frac{9}{2}}$	Homogeneous	$\frac{3(\mu+\alpha)+7}{3}$ $\frac{3(\mu+\alpha)+10}{6}$	$\frac{3(\mu+\alpha)+7}{3}$ $\frac{3(\mu+\alpha)+10}{6}$	Motion is vibratory if $\frac{c_p}{c_v}>\frac{4}{3}$	Anharmonic pulsation of a gas sphere as obtained by Bhatnagar [3] is possible for suitable opacity and energy generation with radiative heat transfer.

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