Chapter 1

Introduction

The main objective of the thesis is to obtain superconvergence results for linear and nonlinear Fredholm integral equations of the second kind. We would like to concentrate on Galerkin and collocation methods. For this purpose, one may use either piecewise or global polynomial basis functions. The numerical solution of Fredholm linear and nonlinear integral equations with piecewise polynomial basis functions and their superconvergence results were studied by many authors (cf. Atkinson and Potra [1], Atkinson and Potra [2], Atkinson [3], Atkinson [5], Chen et al [14], Kaneko and Xu [29], Kaneko et al [30]). However to obtain more accurate solution one has to solve the large linear or nonlinear system of equations which is very computationally expensive. Since, low degree polynomials imply small system of equations, which is highly desirable in practical computations, we employ global polynomial basis functions. In particular, Legendre polynomial basis functions can be generated recursively with ease and possess nice property of orthogonality. Further, these Legendre functions are less expensive computationally compared to piecewise polynomial basis functions, we choose to use these Legendre polynomials as basis functions. However, if $\mathcal{P}_n$ denotes either orthogonal or interpolatory projection from $\mathbb{X}$ into a subspace of global polynomials of degree $\leq n$, then $\|\mathcal{P}_n\|_\infty$ is unbounded. Therefore, there exists at least one $f \in C[-1,1]$ such that $\mathcal{P}_n f \nrightarrow f$. It is the purpose of this work to obtain similar convergence rates for the approximate solutions using Legendre polynomial bases as in the case of spline bases. For this purpose, below we present a brief literature.

Let $\mathbb{X}$ be a Banach space. We consider following integral equations of the form

$$u - \mathcal{K}u = f \tag{1.1}$$

where $\mathcal{K}$ is a linear or nonlinear integral operator. If $\mathcal{K}$ is a linear integral operator and it is defined by

$$\mathcal{K}u(t) = \int_{-1}^{1} k(t,s)u(s)ds, \quad t \in [-1,1],$$

then equation (1.1) reduces to a Fredholm integral equation of the second kind. Numerical solutions of integral equations of the type (1.1), which are reformulations of boundary value problems of partial differential equations has drawn much attention (see, Golberg and Chen [20] and Atkinson [5]). Spline methods and their iterated versions for linear Fredholm integral equations of the second kind have been studied extensively by many researchers (see Sloan [44], Joe [24], Graham et al. [22], Chen and Xu [10]). The Galerkin, collocation, petrov-Galerkin, degenerate kernel and Nyström methods are commonly used methods for approximation of the Fredholm integral equations. The analysis for the convergence of these methods was well documented in Golberg and Chen [20] and Atkinson [5]. Faster convergence of the iterated Galerkin method for Fredholm integral equations of the second kind was first observed by Sloan [43]. On the other hand, the superconvergence of the iterated collocation method was studied in Graham et al. [22] and Joe [24]. The general framework for petrov-Galerkin and iterated petrov-Galerkin methods for integral equations of the second kind was discussed in Chen and Xu [10]. Discrete petrov-Galerkin and discrete wavelet petrov-Galerkin methods for weakly singular integral equations were given in Chen et al. [11], [12]. Fast collocation methods for integral equations of the second kind were discussed in Chen et al. [13]. Multiwavelet applications to the petrov-Galerkin method were given in Chen et al. [9]. Multi-projection methods for the approximation of the equation (1.1) using spline bases were discussed in Kulkarni [36] and Chen et al.

[14], and obtained superconvergence results over iterated Galerkin and iterated collocation methods.

Next let us consider nonlinear integral equation of Hammerstein type which is of the form

$$x - \mathcal{K}\psi x = f, \tag{1.2}$$

where $\mathcal{K}\psi(x)$ is defined by $(\mathcal{K}\psi)(x)(t) = \int_{-1}^{1} k(t,s)\psi(s,x(s))ds$, with the kernel k, f and ψ are known functions and x is the unknown function to be determined.

In Kumar and Sloan [39] and Kumar [40], a new type of collocation method was discussed while establishing superconvergence results for Hammerstein equations. Discrete collocation methods for Hammerstein equations were proposed by Kumar [41] and Atkinson and Flores [4]. A degenerate kernel scheme was introduced by Kaneko and Xu [27] for Hammerstein equations. Superconvergence of the iterated Galerkin, iterated collocation and iterated degenerate kernel methods were studied by Kaneko and Xu [29] and Kaneko et al. [30], [32]. Wavelet applications to the petrov-Galerkin method and its iterated version were given in Kaneko et al. [33]. Some recent results on the numerical solutions of the Hammerstein equations are found in a chapter by Kaneko and Noren [31].

Now let us review the nonlinear integral equation of Urysohn type

$$x - \mathcal{K}(x) = f \tag{1.3}$$

where $\mathcal{K}(x)$ is a nonlinear integral operator defined by $\mathcal{K}(x) = \int_{-1}^{1} k(t,s,x(s))ds$, where the kernel k and f are known functions and x is the unknown solution to be determined. Projection methods and its iterated version for nonlinear integral equations were given in Atkinson and Potra [1]. The superconvergence results for itrated solutions and its discrete version for Urysohn integral equations were studied in Atkinson and Potra [2] and Atkinson and Flores [4]. A survey for nonlinear integral equations of Urysohn type can be found in Atkinson [3].

Next we consider the following Hammerstein integral equation of mixed type

$$x(t) - \sum_{i=1}^{m} \int_{-1}^{1} k_i(t,s)\psi_i(s,x(s))ds = f(t), \ -1 \leq t \leq 1 \tag{1.4}$$

where the kernels k_i, f and ψ_i are known functions and x is the unknown function to be determined. Numerical solvability of these type of integral equations were studied by Ganesh and Joshi [17] and a discrete version was discussed in Ganesh and Joshi [16]. More information on these type of integral equations can be found in a book by Joshi and Bose [25].

1.1 Preliminaries:

Let $\mathbb{X}$ be a Banach space and $\mathbb{BL}(\mathbb{X})$ denote the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$. For $\mathcal{T} \in \mathbb{BL}(\mathbb{X})$, we are interested in the solution of the operator equation

$$(I - \mathcal{T})u = f. \tag{1.5}$$

As the above problem, in general, can not be solved exactly, we consider approximate solutions. Let $\mathcal{T}_n$ be a sequence in $\mathbb{BL}(\mathbb{X})$ such that $\|\mathcal{T}_n - \mathcal{T}\| \to 0$, or $\mathcal{T}_n$ is ν-convergent to $\mathcal{T}$, i.e.,

$$\|\mathcal{T}_n\| \leq C, \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}\| \to 0, \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}_n\| \to 0, \ as\ n \to \infty.$$

Theorem 1.1.1 *Let $\mathbb{X}$ be a Banach space and $\mathcal{T}$, $\mathcal{T}_n$ be bounded linear operators on $\mathbb{X}$. If $\mathcal{T}_n$ is norm convergent to $\mathcal{T}$ or $\mathcal{T}_n$ is ν-convergent to $\mathcal{T}$, and $(I - \mathcal{T})^{-1}$ exists and bounded on $\mathbb{X}$, then $(I - \mathcal{T}_n)^{-1}$ exists and uniformly bounded on $\mathbb{X}$.*

Then the equation (1.5) is approximated by

$$(I - \mathcal{T}_n)u_n = f \ or \ f_n, \tag{1.6}$$

and its iterated solution is defined by $\tilde{u}_n = \mathcal{T}u_n + f$.

The error bounds are given by

$$\|u - u_n\| \leq C\|(\mathcal{T} - \mathcal{T}_n)u\|,$$
$$\|u - \tilde{u}_n\| \leq C\|\mathcal{T}(\mathcal{T} - \mathcal{T}_n)u\|,$$

where C is a constant independent of n.

Banach contraction principle: Let (Ω, d) be a non-empty complete metric space. Let $\mathcal{F} : \Omega \to \Omega$ be a contraction mapping on Ω, i.e. there is a nonnegative real number $q < 1$ such that for all $x, y \in \Omega$

$$d(\mathcal{F}x, \mathcal{F}y) \leq qd(x, y).$$

Then the map $\mathcal{F}$ admits one and only one fixed point x^* in Ω i.e., $\mathcal{F}(x^*) = x^*$.

Lipschitz condition: Let (Ω_1, d_1) and (Ω_2, d_2) be two metric spaces. Then a mapping $\mathcal{F} : \Omega_1 \to \Omega_2$ is said to satisfy a Lipschitz condition with a Lipschitz constant $k \geq 0$ if

$$d_2(\mathcal{F}x, \mathcal{F}y) \leq kd_1(x, y),\ \forall x, y \in \Omega_1.$$

Total variation: The total variation of a real-valued function f defined on an interval $[a, b] \subset \mathbb{R}$ is the quantity

$$V_b^a = \sup_P \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|,$$

where the supremum taken over the set of all partitions $P = \{x_0, x_1, \cdots, x_n\}$ of the given interval $[a, b]$.

Frechet differentiability (Vainikko [48]): Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces and $\mathcal{T} : \mathbb{X} \to \mathbb{Y}$ be an operator (linear or nonlinear). Assume that $\Omega = D(\mathcal{T})$ be an open subset of $\mathbb{X}$. Then $\mathcal{T}$ is said to be Frechet differentiable at $x_0 \in \Omega$, if for any sufficiently small normed element $h \in \mathbb{X}$ such that $x_0 + h \in \Omega$, the increment $\mathcal{T}(x_0 + h) - \mathcal{T}(x_0)$ can be represented in the form

$$\mathcal{T}(x_0 + h) - \mathcal{T}(x_0) = \mathcal{T}'(x_0)h + w(x_0; h),$$

where $\mathcal{T}'(x_0)$ is a linear continuous operator from $\mathbb{X}$ into $\mathbb{Y}$ and $\frac{\|w(x_0;h)\|}{\|h\|} \to 0$, as $\|h\| \to 0$. The linear operator $\mathcal{T}'(x_0)$ is called Frechet derivative of an operator $\mathcal{T}$ at the point x_0.

Note: If $\mathcal{T} : \mathbb{X} \to \mathbb{Y}$ is a linear operator, then $\mathcal{T}'(x) = \mathcal{T}(x)$, $\forall x \in \mathbb{X}$. We also note that the Frechet derivative of an operator $\mathcal{T} : D(\mathcal{T}) \subseteq \mathbb{X} \to \mathbb{Y}$ at $x \in D(\mathcal{T})$ is a bounded linear operator from $\mathbb{X}$ into $\mathbb{Y}$.

Theorem 1.1.2 (Suhubi [46]) *Suppose $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ be Banach spaces and $\mathcal{T}_1 : \mathbb{X} \to \mathbb{Y}$ is Frechet differentiable at x and $\mathcal{T}_2 : \mathbb{Y} \to \mathbb{Z}$ is Frechet differentiable at $\mathcal{T}_1(x)$. Then $H = \mathcal{T}_1 o \mathcal{T}_2$ is Frechet differentiable at x and $H'(x) = (\mathcal{T}_1 o \mathcal{T}_2)'(x) = \mathcal{T}_1'(\mathcal{T}_2(x))(\mathcal{T}_2'(x))$.*

Note: Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and $\mathcal{T} : \mathbb{X} \to \mathbb{Y}$ be a nonlinear operator. If $\mathcal{T}$ is Frechet differentiable at every vector $u + t(w-u)$ with $0 \le t \le 1$ for all $u, w \in \mathbb{X}$, then there exists a number $0 < \delta < 1$ such that

$$\|\mathcal{T}(w) - \mathcal{T}(u)\| \le \sup_{0<\delta<1} \|\mathcal{T}'(u + \delta(w-u))\| \|w-u\|.$$

Analogous to the concept of compactness of a linear operator, a nonlinear operator $\mathcal{T} : D(\mathcal{T}) \subseteq \mathbb{X} \to \mathbb{Y}$ is said to be compact if for every bounded subset $\mathbb{E}$ of $\mathbb{X}$, closure of $\mathcal{T}(\mathbb{E})$, i.e, $\overline{\mathcal{T}(\mathbb{E})}$ is compact set in $\mathbb{Y}$.

Theorem 1.1.3 (Joshi [25]) *Suppose $\mathcal{T} : \mathbb{X} \to \mathbb{Y}$ is compact and continuous and has Frechet derivative $\mathcal{T}'(x)$ at $x \in \mathbb{X}$. Then $\mathcal{T}'(x)$ is also compact and continuous operator.*

Proof. Suppose $\mathcal{T}'(x)$ is not compact. Then there exists $\epsilon > 0$ and a sequence $\{x_n\}$ with $\|x_n\| \le 1$ such that

$$\|\mathcal{T}'(x)x_n - \mathcal{T}'(x)x_m\| > 3\epsilon, \quad m \ne n.$$

But

$$\mathcal{T}(x+h) - \mathcal{T}(x) = \mathcal{T}'(x)h + w(x,h)$$

and

$$\|w(x,h)\| \le \epsilon \|h\| \text{ if } \|h\| < \delta \text{ for some } \delta.$$

Thus we have

$$\begin{aligned}\|\mathcal{T}(x+\delta x_n)-\mathcal{T}(x+\delta x_m)\| &\geq \delta\|\mathcal{T}'(x)(x_n-x_m)\|-\|w(x,\delta x_m)\|-\|w(x,\delta x_n)\| \\ &> 3\delta\epsilon-\delta\epsilon-\delta\epsilon=\delta\epsilon,\end{aligned}$$

which is impossible since $\mathcal{T}$ is compact. Hence $\mathcal{T}'(x)$ is a compact operator.

This completes the proof. □

Note: If $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n$ are nonlinear compact operators on $\mathbb{X}$, then $(\mathcal{T}_1+\mathcal{T}_2+\cdots+\mathcal{T}_n)'(x) = \mathcal{T}_1'(x)+\mathcal{T}_2'(x)+\cdots+\mathcal{T}_n'(x)$ is also compact operator on $\mathbb{X}$.

Let $\hat{\mathcal{T}}$ and $\tilde{\mathcal{T}}$ be continuous operators over an open set Ω in a Banach space $\mathbb{X}$. Let us consider under what conditions the solvability of one of the equations $x=\hat{\mathcal{T}}x$ and $x=\tilde{\mathcal{T}}x$ leads to the solvability of the other.

Theorem 1.1.4 (Vainikko [48]) *Let the equation $x=\tilde{\mathcal{T}}x$ has an isolated solution $\tilde{x}_0 \in \Omega$ and let the following conditions be satisfied.*

(a) The operator $\hat{\mathcal{T}}$ is Frechet differentiable in some nbd of the point $\tilde{x}_0$, while the linear operator $I-\hat{\mathcal{T}}'(\tilde{x}_0)$ is continuously invertible.

(b) Suppose that for some $\delta>0$ and $0<q<1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|x-\tilde{x}_0\|\leq\delta$ is contained within Ω),

$$\sup_{\|x-\tilde{x}_0\|\leq\delta}\|(I-\hat{\mathcal{T}}'(\tilde{x}_0))^{-1}(\hat{\mathcal{T}}'(x)-\hat{\mathcal{T}}'(\tilde{x}_0))\|\leq q, \tag{1.7}$$

$$\alpha=\|(I-\hat{\mathcal{T}}'(\tilde{x}_0))^{-1}(\hat{\mathcal{T}}(\tilde{x}_0))-\tilde{\mathcal{T}}(\tilde{x}_0))\|\leq\delta(1-q). \tag{1.8}$$

Then the equation $x=\hat{\mathcal{T}}x$ has in the sphere $\|x-\tilde{x}_0\|\leq\delta$ a unique solution $\hat{x}_0$. Moreover, the inequality

$$\frac{\alpha}{1+q}\leq\|\hat{x}_0-\tilde{x}_0\|\leq\frac{\alpha}{1-q} \tag{1.9}$$

is valid.

1.2 The structure of the Thesis:

The research work in this thesis has been organized into four chapters (Chapters: 2 to 5), which deals with a relevant contribution of the author in the field of integral equations. A summary outline of the same is described as follows:

In **Chapter 2**, we describe the numerical solution of Fredholm integral equations of the second kind (1.1) with a smooth kernel using multi-projection method. We consider the approximating space $\mathbb{X}_n$ as Legendre polynomial subspaces of degree $\leq n$. We obtain superconvergence results for the approximate solutions in Legendre M-Galerkin and Legendre M-collocation methods in both L^2-norm and infinity norm. We prove that in iterated Legendre M-Galerkin methods not only iterative solution $\tilde{u}_n^{M,G}$ approximates the exact solution u in the supremum norm with the order of convergence n^{-4r}, but also the derivative of $\tilde{u}_n^{M,G}$ approximates the corresponding derivative of u in the infinity norm with the same order of convergence, where r denotes the smoothness of the kernel. We also obtain superconvergence rates for the approximate solutions in case of iterated Legendre M-collocation method in both L^2-norm and infinity norm.

In **Chapter 3**, we approximate the solution of nonlinear integral equations of Hammerstein type (1.2) with a smooth kernel. We consider the approximating space $\mathbb{X}_n$ as Legendre polynomial subspaces of degree $\leq n$. We discuss Legendre Galerkin and Legendre collocation methods for solving the Hammerstein integral equations and we obtain convergence rates for the approximate solutions in both L^2-norm and infinity norm. We obtain superconvergence results for iterated Legendre Galerkin solution in both L^2-norm and infinity norm. Iterated Legendre collocation solution gives better convergence rates over the Legendre collocation solution in infinity norm. We prove that iterated Legendre Galerkin method improves over iterated Legendre collocation method for Hammerstein integral equations.

In **Chapter 4**, we consider the approximate solution of nonlinear integral equations of Urysohn type (1.3) with a smooth kernel. We consider the approximating space $\mathbb{X}_n$ as Legendre polynomial subspaces of degree $\leq n$. We obtain the similar superconvergence rates as in the case of Hammerstein integral equations in chapter−3.

In **Chapter 5**, we approximate the solution of nonlinear Hammerstein integral equations of mixed type (1.4) with a smooth kernel. We consider the approximating space $\mathbb{X}_n$ as Legendre polynomial subspaces of degree $\leq n$. We discuss Legendre Galerkin and Legendre collocation methods for solving the Hammerstein integral equations of mixed type and we obtain convergence rates for the approximate solutions in both L^2-norm and infinity norm. As in chapter−3, we obtain superconvergence results for iterated Legendre Galerkin solution in both L^2-norm and infinity norm. Iterated Legendre collocation solution gives better convergence rates over the Legendre collocation solution in infinity norm. We prove that iterated Legendre Galerkin method improves over iterated Legendre collocation method for these Hammerstein integral equations of mixed type.

1.3 Contributions of the Thesis:

In this thesis, we consider numerical solutions of linear and nonlinear Fredholm integral equations. We propose multi-projection and iterated multi-projection methods for Fredholm integral equations of the second kind with a smooth kernel using Legendre polynomial bases. We obtain superconvergence results for the approximate solutions. More precisely, we prove that in Legendre M-Galerkin and Legendre M-collocation methods not only the iterative solution $\tilde{u}_n$ approximates the exact solution u in the infinity norm with the order of convergence n^{-4r}, but also the derivatives of $\tilde{u}_n$ approximate the corresponding derivatives of u in the infinity norm with the same order of convergence. Next we consider the Galerkin and collocation methods for solving the nonlinear integral equations of Hammerstein and Urysohn type with smooth kernels using Legendre polynomial bases. We obtain convergence rates for the approximate solutions in both L^2-norm and infinity norm. Also we obtain superconvergence results for the iterated Legendre

Galerkin solutions in both L^2-norm and infinity norm. We prove that the iterated Legendre Galerkin method improves over the iterated Legendre collocation method for both Hammerstein and Urysohn integral equations. We extend these results to the Hammerstein integral equations of mixed type with a smooth kernel using Legendre polynomial basis functions and obtain similar superconvergence results for the approximate solutions in both L^2-norm and infinity norm.

References

[1] Atkinson, K. and Potra, F. Projection and iterated projection methods for nonlinear integral equations, SIAM J. Numer. Anal. 24 (1987)(6): 1352-1373.

[2] Atkinson, K. and Potra, F. The discrete Galerkin method for nonlinear integral equations, J. Integral Equations Appl. 1 (1988) (1): 17-54.

[3] Atkinson, K. A survey of numerical methods for solving nonlinear integral equations, J. Integral Equations Appl. 4 (1992) (1): 15-46.

[4] Atkinson, K. and Flores, J. The discrete collocation method for nonlinear integral equations, IMA J. Numer. Anal. 13 (1993) (2): 195-213.

[5] Atkinson, K. E. The numerical solution of integral equations of the second kind, Cambridge Monographs on Applied and Computational Mathematics, 4. Cambridge University Press, Cambridge, 1997.

[6] Baker, C. T. H. The numerical treatment of integral equations, Monographs on Numerical Analysis. Clarendon Press, Oxford, 1977.

[7] Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zang, T. A. Spectral methods, fundamentals in single domains. Scientific Computation. Springer-Verlag, Berlin, 2006.

[8] Chatelin, F. Spectral approximation of linear operators, Academic Press, New York, 1983.

[9] Chen, Z., Micchelli, C. A. and Xu, Y. The petrov-Galerkin method for second kind integral equations. II. multiwavelet schemes, Adv. Comput. Math. 7 (1992) (3): 199-233.

[10] Chen, Z. and Xu, Y. The petrov-Galerkin and iterated petrov-Galerkin methods for second kind integral equations, SIAM J. Numer. Anal. 35 (1998) (1): 406-434.

[11] Chen, Z., Xu, Y. and Zhao, J. The discrete petrov-Galerkin method for weakly singular integral equations, J. Integral Equations Appl. 11 (1999) (1): 1-35.

[12] Chen, Z., Micchelli, C. A. and Xu, Y. Discrete wavelet petrov-Galerkin methods, Adv. Comput. Math. 16 (2002) (1): 1-28.

[13] Chen, Z., Micchelli, C. A. and Xu, Y. Fast collocation methods for second kind integral equations, SIAM J. Numer. Anal. 40 (2002) (1): 344-375.

[14] Chen, Z., Long, G. and Nelakanti, G. The discrete multi-projection method for Fredholm integral equations of the second kind, J. Integral Equations Appl. 19 (2007) (2): 143-162.

[15] Fang, W., Wang, Y., and Xu, Y. An implementation of fast wavelet Galerkin methods for integral equations of the second kind, J. Sci. Comput. 20 (2004) (2): 277-302.

[16] Ganesh, M. and Joshi, M. C. Discrete numerical solvability of Hammerstein integral equations of mixed type, J. Integral Equations Appl. 2 (1989) (1): 107-124.

[17] Ganesh, M. and Joshi, M. C. Numerical solvability of Hammerstein integral equations of mixed type, IMA J. Numer. Anal. 11 (1991) (1): 21-31.

[18] Golberg, M. A. Discrete polynomial-based Galerkin methods for Fredholm integral equations, J. Integral Equations Appl. 6 (1994) (2): 197-211.

[19] Golberg, M. Improved convergence rates for some discrete Galerkin methods, J. Integral Equations Appl. 8 (1996) (3): 307-335.

[20] Golberg, M. A. and Chen, C. S. Discrete projection methods for integral equations. Computational Mechanics Publications, Southampton, 1997.

[21] Graham, I. G. Galerkin methods for second kind integral equations with singularities, Math. Comp. 39 (1982) (160): 519-533.

[22] Graham, I. G., Joe, S. and Sloan, I. H. Iterated Galerkin versus iterated collocation for integral equations of the second kind, IMA J. Numer. Anal. 5 (1985) (3): 355-369.

[23] Hackbush, W. Integral equations. Theory and numerical treatment, International Series of Numerical Mathematics, 120. Birkhäuser Verlag, Basel, 1995.

[24] Joe, S. Collocation methods using piecewise polynomials for second kind integral equations, J. Comp. Appl. Math. 12-13 (1985) : 391-400.

[25] Joshi, M. C. and Bose, R. K. Some topics in nonlinear functional analysis, John Wiley and Sons Inc, 1985.

[26] Kaneko, H., Noren, R. D. and Xu, Y. Regularity of the solution of Hammerstein equations with weakly singular kernels, Int. Eqs. Oper. Theory. 13 (1990) (5): 660-670.

[27] Kaneko, H. and Xu, Y. Degenerate kernel method for Hammerstein equations, Math. Comp. 56 (1991) (193): 141-148.

[28] Kaneko, H., Noren, R. D. and Xu, Y. Numerical solutions for weakly singular Hammerstein equations and their superconvergence, J. Integral Equations Appl. 4 (1992) (3): 391-407.

[29] Kaneko, H. and Xu, Y. Superconvergence of the iterated Galerkin methods for Hammerstein equations, SIAM J. Numer. Anal. 33 (1996) (3): 1048-1064.

[30] Kaneko, H., Noren, R. D. and Padilla, P. A. Superconvergence of the iterated collocation methods for Hammerstein equations, J. Comput. Appl. Math. 80 (1997) (2): 335-349.

[31] Kaneko, H. and Noren, R. D. Numerical solutions of Hammerstein equations. Boundary integral methods: numerical and mathematical aspects, 257-288, Comput. Eng., 1, WIT Press/Comput. Mech. Publ., Boston, MA, 1999.

[32] Kaneko, H., Padilla, P. and Xu, Y. Superconvergence of the iterated degenerate kernel method, Appl. Anal. 80 (2001) (3-4): 331-351.

[33] Kaneko, H., Noren, R. D. and Novaprateep, B. Wavelet applications to the petrov-Galerkin method for Hammerstein equations, Appl. Numer. Math. 45 (2003) (2-3): 255-273.

[34] Krasnoselskii, M. A. Topological methods in the theory of nonlinear integral equations. Translated by A. H. Armstrong; translation edited by J. Burlak. Pergamon Press, The Macmillan Co., New York, 1964.

[35] Krasnoselski, M. A., Vanikko, G. M., Zabreko, P. P., Rutitskii, Y. B. and Stetsenko, V. Y. Approximate solution of operator equations. Translated from the Russian by D. Louvish, Wolters-Noordhoff Publishing, Groningen, 1972 .

[36] Kulkarni, R. P. A superconvergence result for solutions of compact operator equations, Bull. Austral. Math. Soc. 68 (2003) (3): 517-528.

[37] Kulkarni, R. P. A new superconvergent projection method for approximate solutions of eigenvalue problems, Numer. Funct. Anal. Optim. 24 (2003) (1-2): 75-84.

[38] Kulkarni, R. P. A new superconvergent collocation method for eigenvalue problems, Math. Comp. 75 (2006) (254): 847-857.

[39] Kumar, S., Sloan, I. A new collocation-type method for Hammerstein integral equations, Math. of Comp 48 (1987) : 585-593.

[40] Kumar, S. Superconvergence of a collocation-type method for Hammerstein equations, IMA J. Numer. Anal. 7 (1987) (3): 313-325.

[41] Kumar, S. A discrete collocation-type method for Hammerstein equations, SIAM J. Numer. Anal. 25 (1988) (2): 328-341.

[42] Kumar, S. The numerical solution of Hammerstein equations by a method based on polynomial collocation, J. Austral. Math. Soc. Ser. B 31 (1990) (3): 319-329.

[43] Sloan, I. H. Improvement by iteration for compact operator equations, Math. Comp. 30 (1976) (136): 758-764.

[44] Sloan, I. H. Four variants of the Galerkin method for integral equations of the second kind, IMA J. Numer. Anal. 4 (1984) (1): 9-17.

[45] Sloan, I. H. and Thomee, V. Superconvergence of the Galerkin iterates for integral equations of the second kind, J. Integral Equations 9 (1985) (1): 1-23.

[46] Suhubi, Erdogan. S. Functional analysis, Kluwer Academic Publishers, Boston, London, 2003.

[47] Tang, Tao., Xu, Xiang. and Cheng, Jin. On spectral methods for volterra integral equations and the convergence analysis, J. computational. Math. 26 (2008) (6): 825-837.

[48] Vainikko, G. M. A perturbed Galerkin method and the general theory of approximate methods for nonlinear equations, USSR Computational Mathematics and Mathematical physics 7 (1967) (4): 1-41.

[49] Vainikko, G. M. and Karma, O. The convergence of approximate methods for solving linear and non-linear operator equations, USSR Computational Mathematics and Mathematical physics 14 (1974) (4): 9-19.

[50] Weiss, R. On the approximation of fixed points of nonlinear compact operators, SIAM J. Numer. Anal. 11 (1974) (3): 550-553.

[51] Zabreko, P. Integral equations: a reference text, Noordhoff International Publishing, 1975.

[52] Zeidler, E. Nonlinear functional analysis and its applications. II/B, Springer-Verlag, New York, 1990.