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A

THESIS ON

'ROTATIONAL FLOWS'

By

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CERTIFICATE

This is to certify that the thesis entitled 'Rotational Flows' that is being submitted by Sri L.V.K.Viswanadha Sarma for the award of the Degree of Doctor of Philosophy to the Indian Institute of Technology, Kharagpur is a record of bonafide research work carried out by him under my supervision and guidance. Sri Sarma has worked for two years in the Department of Applied Mathematics, Indian Institute of Technology, Kharagpur and the thesis has reached the standard fulfilling the requirements of the regulations to the Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

This thesis deals with some problems in rotational flows. It is divided into five chapters.

The motion of a solid body in a fluid endowed with vorticity is not, in general, easily amenable to mathematical treatment. But some progress has been made in recent years in both two and three-dimensional rotational flows. When the motion is two-dimensional and the vorticity uniform, the exact steady state equations for an inviscid, incompressible fluid can be reduced to one Laplace equation for the stream function, which is separable in a number of coordinate systems. When the method of separation of variables is not applicable, the method of conformal transformation may be applied. In chapter I the latter method is used to solve the problem of the rotational flow of a liquid past a cylinder of regular polygonal cross-section. The stream function for the flow is obtained and the thrust on the cylinder due to liquid motion is calculated. It is found that the thrust is independent of the number of sides of the polygon when the length of a side is taken as $2\, d_n \sin(\pi/n)$, d_n being the radius of the circumscribed circle and n the number of sides of the polygon. Further, the corresponding problem for hypotrochoidal cylinders is also considered and the thrust is calculated in each case. The results for cross-sections of equilateral triangle and square are compared with those of hypotrochoids of three and four cusps respectively.

The rest of the work deals with three-dimensional rotational flows with axial symmetry. The motion of bodies in a uniformly rotating fluid has been considered by various authors. Inspite of the simplifying assumptions of an incompressible and inviscid fluid, not much progress has been made in attempting to explain some of the experimental results theoretically. The solution of the exact steady state equations has been found to be indeterminate due to lack of sufficient boundary conditions. One possible reason for this indeterminacy lay perhaps in the manner in which the flow was started. Further, even if a steady solution of the governing equation can be obtained, there is no guarantee, in this type of fluid motion, that the flow can be set up by starting the body from rest relative to the rotating system. Keeping this in view, various attempts have been made in recent years to obtain some general features of the flow on the basis of the linearized equations. In chapter II of this dissertation the slow uniform motion, after an impulsive start from relative rest, of a paraboloid of revolution along the axis of a rotating liquid is investigated by using a perturbation method. The principal purpose is to explain the mechanism by which the fluid is not subjected to any substantial radial displacement, which is a direct consequence of the requirement that the circulation round material circuits should be constant when the perturbation velocities remain small. It appears that the mechanism is an oscillatory one in which the distance between any fluid particle and the axis of rotation oscillates sinusoidally in time with small amplitude. As time progresses, the amplitude of the oscillation decays to zero everywhere except on the paraboloid. The ultimate motion is then a rigid body rotation everywhere except on the body and the axis of rotation, where the perturbation velocities continue to oscillate indefinitely with small amplitude.

In chapter III the flow of a rotating liquid past a sphere in a cylindrical pipe is investigated on the basis of the linearized equations. Assuming the pipe radius to be large compared to the radius of the sphere, it is found that the flow in this case is also oscillatory in character. Further, there are no disturbances far upstream and waves are propagated downstream only. The ultimate flow is steady and two-dimensional.

In chapter IV the oscillation of an oblate spheroid in a rotating fluid is considered. The case of a circular disc, with its centre on the axis of rotation and its plane perpendicular to this axis, which oscillates in a direction normal to its surface, is deduced from that of the oblate spheroid.

The basic problem in the flow of rotating fluid is to study the interaction between the rotational motion and the motion in meridian planes. These two motions are not independent; changes in the angular velocity affect the centrifugal force and lead to motions in meridian planes and these in turn affect the rotation through the Coriolis force. In the case of inviscid fluid these two motions are related in a simple manner. But when viscosity is taken into account the relation is so complicated that even the linearized equations appear to be intractable for exact solution. In chapter V the slow uniform motion of a sphere, started impulsively from rest, in a viscous rotating fluid is considered. A solution valid for small values of the Reynold's number Ua/ν, U being the uniform velocity of the sphere, a the radius of the sphere and ν the kinematic coefficient of viscosity of the fluid, is obtained. It is shown that the flow tends to a steady state ultimately. The fluid resistance on the sphere is calculated and it is found that the effect of rotation of the fluid is to further increase the resistance on the sphere.

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CHAPTER I

ROTATIONAL FLOW OF A LIQUID PAST A REGULAR POLYGONAL CYLINDER+

1.1. Introduction

The motion of a solid in a liquid possessing vorticity is a problem of considerable interest. The stream function ψ for the steady, two-dimensional rotational flow of an incompressible inviscid fluid satisfies

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \omega = 0 , \tag{1.1.1}$$

where the vorticity ω , given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} , \tag{1.1.2}$$

satisfies the equation

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0 \tag{1.1.3}$$

where

$$u = \frac{\partial \psi}{\partial y} , \quad v = -\frac{\partial \psi}{\partial x} ,$$

+ Published in the 'Proceedings of the Indian Academy of Sciences', Vol.XLVI, No.3, Sec.A, 1957, p.224.

and u, v are the velocity components of the fluid parallel to the x-axis and y-axis respectively.

The general solution of (1.1.3) is

$$\omega = f(\psi) \tag{1.1.4}$$

and so the vorticity is constant along a stream line. A few exact solutions of (1.1.1) and (1.1.4) are known when $f(\psi)$ is equal to a constant and when $f(\psi) = K\psi$, K being a constant. Recently, some interesting solutions of (1.1.1) and (1.1.4) are obtained by Seth[21] when $f(\psi) = \exp(\psi)$ and when $f(\psi) = \psi^2$. Also a number of investigations using approximate methods have been carried out by various authors[7],[12],[15],[16].

When $f(\psi)$ is a constant, equation (1.1.1) may be transformed into Laplace equation which is separable in a number of coordinate systems. G.I.Taylor[25] has discussed the motion of a circular cylinder under the assumption that the undisturbed motion of the fluid consists of a uniform rotation Ω about the origin, so that the vorticity ω is equal to 2Ω. The corresponding problem for the motion of an elliptic cylinder has been treated by M.Roy[19]. These authors show that for a cylinder moving in any manner, in general two forces X and Y and a couple are necessary to maintain the motion.

The interest in the field has been recently revived by a number of papers by Tsien[29], Richardson[18], A.R.Mitchell and J.D.Murray[11], and De[1],[2]. A.R.Mitchell and J.D. Murray[11] have investigated in detail the flow past circular, elliptic and parabolic cylinders by the method of separation of variables. By an extension of Blasius' Theorem for rotational flow, De[2] calculated the thrust due to the flow past cylinders whose cross-sections are bounded by certain family of curves.

The present investigation deals with the rotational flow past a cylinder of regular polygonal cross-section. The problem is simplified by transforming the area outside the polygon into the interior of a unit circle. The thrust on the cylinder due to liquid motion is calculated and it is found that the thrust is independent of the number of sides of the polygon provided the length of a side is taken as $2 d_n \sin(\pi/n)$, d_n being the radius of the circumscribed circle and n the number of sides of the polygon. d_n, 'qua' a function of n, is given in (1.3.6) below. The thrust for unit area of cross-section of the cylinder is found to decrease as the number of sides of the polygon increases. Further, the corresponding problem for hypotrochoidal cylinders is considered and the thrust is calculated in each case. The results for cross-sections of equilateral triangle and square are compared with those of hypotrochoids of three and four cusps respectively.

1.2. Formulation of the problem

Let the undisturbed motion of the fluid consist of uniform velocity U along the x-axis and uniform vorticity ω, so that the stream function ψ_2 of the undisturbed flow is given by

$$\psi_2 = Uy - \tfrac{1}{2}\omega y^2 . \tag{1.2.1}$$

The stream function ψ_1 for the disturbed flow satisfies the equation

$$\nabla^2 \psi_1 + \omega = 0 . \tag{1.2.2}$$

Putting $\psi = \psi_1 - \psi_2$, we have

$$\nabla^2 \psi = 0 . \tag{1.2.3}$$

The boundary conditions are that $\psi \longrightarrow 0$ at infinity and

$$\psi = -Uy + \tfrac{1}{2}\omega y^2 \tag{1.2.4}$$

over the boundary.

1.3. Conformal transformation

Taking the centre of the polygon as origin in the z-plane, the area outside the polygon can be transformed into the interior of a unit circle in the t-plane by means of the relation[20]

$$\frac{dz}{dt} = \frac{A \prod_{r=1}^{n} (t - t_r)^{2/n}}{t^2}, \tag{1.3.1}$$

where $t_1, t_2, \cdots t_n$ correspond to the n vertices of the polygon and lie on the circle $|t| = 1$. The symmetry of the figure shows that we can take the t's to be the n roots of $t^n = 1$. Thus (1.3.1) takes the simple form

$$\frac{dz}{dt} = \frac{A(1-t^n)^{2/n}}{t^2}. \tag{1.3.2}$$

By adjusting the constant A, we can write

$$z = \frac{1}{t}(1-t^n)^{2/n} + 2\int_0^t t^{n-2}(1-t^n)^{\frac{2}{n}-1}\,dt \tag{1.3.3}$$

$$= \frac{1}{t} + \sum_{r=1}^{\infty} a_{rn-1}\, t^{rn-1}, \tag{1.3.4}$$

where

$$a_{n-1} = \frac{2}{n(n-1)},$$

$$a_{rn-1} = \frac{2}{n}\left(1-\frac{2}{n}\right)\left(2-\frac{2}{n}\right)\cdots\cdots\left(r-1-\frac{2}{n}\right)\frac{1}{\lfloor r \;(rn-1)}.$$

The infinite series on the right of (1.3.4) is absolutely convergent for $|t| \leq 1$. The radius of the circumscribed circle is given by

$$d_n = 1 + \sum_{r=1}^{\infty} a_{rn-1}\,, \tag{1.3.5}$$

and a side of the polygon is

$$2\, d_n \sin(\pi/n)\,.$$

Using (1.3.3), we get

$$d_n = 2\int_0^1 t^{n-2}(1-t^n)^{\frac{2}{n}-1}\,dt$$

$$= \frac{2}{n}\sin\frac{\pi}{n}\,\Gamma\left(\frac{2}{n}\right)\left\{\Gamma\left(1-\frac{1}{n}\right)\right\}^2. \tag{1.3.6}$$

For $n = 3, 4$, we find

$$d_3 = 1.3692\,,$$
$$d_4 = 1.1982\,. \tag{1.3.7}$$

1.4. Solution to the problem

From (1.2.4) we have on the boundary

$$\psi = -Uy + \tfrac{1}{2}\omega y^2 + c$$

$$= -\frac{U}{2i}(z-\bar{z}) - \tfrac{1}{8}\omega(z-\bar{z})^2 + c, \tag{1.4.1}$$

where $\bar{z}$ is the complex conjugate of z.

From (1.3.4) we have

$$z = \sum_{r=0}^{\infty} a_{rn-1} t^{rn-1}, \quad (a_{-1} = 1),$$

$$(z-\bar{z})^2 = \left(\frac{1}{t} - \frac{1}{\bar{t}}\right)^2 + \sum_{r=1}^{\infty} a_{rn-1}^2\left(t^{2rn-2} + \bar{t}^{2rn-2} - 2\right) +$$

$$+ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rn-1} a_{sn-1}\left\{t^{rn+sn-2} + \bar{t}^{rn+sn-2} - \right.$$

$$\left. - \left(t^{rn-sn} + \bar{t}^{rn-sn}\right)\right\}, \tag{1.4.2}$$

$$r \neq s,$$

where $\bar{t}$ is the complex conjugate of t.

Now

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rn-1} a_{sn-1} t^{rn-sn} = \sum_{m=1}^{\infty} K_m\left(t^{nm} + t^{-nm}\right); \tag{1.4.3}$$

where

$$K_m = \sum_{r=0}^{\infty} a_{rn-1}\, a_{rn+mn-1} \qquad (1.4.4)$$

Substituting these in (1.4.1) we get

$$\psi = -\frac{U}{2i}\left[\left(\frac{1}{t}-\frac{1}{\bar{t}}\right)+\sum_{r=1}^{\infty} a_{rn-1}\left(t^{rn-1}-\bar{t}^{\,rn-1}\right)\right]-$$

$$-\frac{1}{8}\omega\left[\left(\frac{1}{t}-\frac{1}{\bar{t}}\right)^2+\sum_{r=1}^{\infty} a^2_{rn-1}\left(t^{2rn-2}+\bar{t}^{\,2rn-2}\right)+\right.$$

$$+\sum_{0}^{\infty}\sum_{0}^{\infty} a_{rn-1}\, a_{sn-1}\left(t^{rn+sn-2}+\bar{t}^{\,rn+sn-2}\right)-$$

$$\left.-2\sum_{m=1}^{\infty} K_m\left(t^{nm}+\bar{t}^{\,nm}\right)\right]+C. \qquad (1.4.5)$$

Taking (R, θ) as the polar coordinates in the t-plane, we can write on the boundary of the unit circle $R = 1$,

$$\psi = -U\left[-\sin\theta + \sum_{r=1}^{\infty} a_{rn-1}\sin(rn-1)\theta\right] -$$

$$-\frac{1}{4}\omega\left[\cos 2\theta + \sum_{r=1}^{\infty} a^2_{rn-1}\cos(2rn-2)\theta + \right.$$

$$+\sum_{0}^{\infty}\sum_{0}^{\infty} a_{rn-1}a_{sn-1}\cos(rn+sn-2)\theta -$$

$$\left. -2\sum_{m=1}^{\infty} K_m \cos nm\theta\right] + C. \tag{1.4.6}$$

Therefore we take ψ to be the imaginary part of f where

$$f = U\left[t - \sum_{r=1}^{\infty} a_{rn-1}t^{rn-1}\right] -$$

$$-\frac{1}{4}i\omega\left[t^2 + \sum_{r=1}^{\infty} a^2_{rn-1}t^{2rn-2} + \right.$$

$$\left. + \sum_{0}^{\infty}\sum_{0}^{\infty} a_{rn-1}a_{sn-1}t^{rn+sn-2} - 2\sum_{1}^{\infty} K_m t^{nm}\right]. \tag{1.4.7}$$

Since

$$z^2 = \frac{1}{t^2} + \sum_{r=1}^{\infty} a^2_{rn-1}t^{2rn-2} +$$

$$+ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rn-1}a_{sn-1}t^{rn+sn-2},$$

we can write

$$f = U\left[-z + \left(t + \frac{1}{t}\right)\right] -$$

$$-\frac{1}{4} i\omega\left[z^2 + t^2 - \frac{1}{t^2} - 2\sum_{m=1}^{\infty} K_m t^{nm}\right]. \quad (1.4.8)$$

Therefore

$$\psi = -Uy + U\left(R - \frac{1}{R}\right)\sin\theta -$$

$$-\frac{1}{4}\omega\left[x^2 - y^2 + \left(R^2 - \frac{1}{R^2}\right)\cos 2\theta -\right.$$

$$\left. - 2\sum_{m=1}^{\infty} K_m R^{nm} \cos nm\theta\right]. \quad (1.4.9)$$

Since on the boundary $R = 1$ and

$$x^2 + y^2 = z\bar{z}$$

$$= \sum a^2_{kn-1} + 2\sum_{m=1}^{\infty} K_m \cos nm\theta,$$

the boundary condition is satisfied.

For the flow past the cylinder we get the stream function

$$\psi_1 = U\left(R - \frac{1}{R}\right)\sin\theta - \frac{1}{4}\omega\left[x^2 + y^2 +\right.$$

$$\left. + \left(R^2 - \frac{1}{R^2}\right)\cos 2\theta - 2\sum_{m=1}^{\infty} K_m R^{nm} \cos nm\theta\right]. \quad (1.4.10)$$

1.5. Liquid pressure on the cylinder

Let X , Y be the components of the resultant thrust of the liquid on the cylinder along the axes and M the moment about the origin. If l, m be the direction cosines of the outward drawn normal to the boundary of the cylinder and p be the pressure, then

$$X = -\int p\, l\, ds$$

$$= -\int p\, dy \,,$$

$$Y = -\int p\, m\, ds$$

$$= \int p\, dx \,,$$

so that

$$X - iY = -i\int p\, d\bar{z}$$

and

$$M = \int p\,(mx - ly)\, ds$$

$$= \int p\,(x\, dx + y\, dy)\,,$$

the integrals being all taken round the contour of the cylinder. Now if q is the velocity, the pressure p at any point is given by

$$p = K - \tfrac{1}{2}\rho q^2 \,,$$

where ρ is density of the liquid and K is constant along a stream line. Since the boundary of the cross-section is a stream line, K is constant on the boundary. So we get

$$X - iY = \frac{1}{2} i \rho \int q^2 \, dz$$

and

$$M = -\frac{1}{2} \rho \int q^2 (x\,dx + y\,dy)$$

$$= \text{real part of } -\frac{1}{2} \rho \int q^2 z \, d\bar{z} .$$

Now since

$$u = \frac{\partial \psi}{\partial y} , \qquad v = -\frac{\partial \psi}{\partial x} ,$$

we have

$$(u + iv)\, d\bar{z} = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy - i\, d\psi ,$$

$$(u - iv)\, dz = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy + i\, d\psi .$$

On the boundary C of the cylinder $d\psi = 0$, so that on C

$$(u + iv)\, d\bar{z} = (u - iv)\, dz .$$

Therefore

$$q^2 \, d\bar{z} = (u + iv)(u - iv) \, d\bar{z}$$

$$= (u - iv)^2 \, dz \, .$$

Hence

$$X - iY = \tfrac{1}{2} i \rho \int_C (u - iv)^2 \, dz \qquad (1.5.1)$$

and

$$M = \text{real part of } -\tfrac{1}{2} \rho \int_C z (u - iv)^2 \, dz \, . \qquad (1.5.2)$$

The above formulae are extensions[1] to the case of rotational flow of the corresponding formulae of Blasius for irrotational flow[8].

From (1.4.10) the stream function for the flow past the cylinder is the imaginary part of

$$U\left(t + \frac{1}{t}\right) - \frac{1}{4} i \omega \left[x^2 + y^2 + t^2 - \frac{1}{t^2} - 2 \sum_{m=1}^{\infty} K_m t^{nm} \right] .$$

The complex velocity at any point is therefore given by

$$u - i v = \tfrac{1}{2} \cdot i \omega \bar{z} - \Big[U\left(1 - \tfrac{1}{t^2}\right) -$$

$$- \tfrac{1}{2} i \omega \Big(t + \tfrac{1}{t^3} - n \sum_{m=1}^{\infty} K_m \, m \, t^{nm-1} \Big) \Big] \frac{dt}{dz}. \tag{1.5.3}$$

On the boundary

$$u - i v = \tfrac{1}{2} i \omega \sum_{r=0}^{\infty} a_{rn-1} \, t^{-rn+1} -$$

$$- \Big[U\left(1 - \tfrac{1}{t^2}\right) - \tfrac{1}{2} i \omega \Big(t + \tfrac{1}{t^3} - n \sum_{1}^{\infty} K_m m t^{nm-1} \Big) \Big] \frac{dt}{dz}. \tag{1.5.4}$$

Putting

$$G = \tfrac{1}{2} i \omega \sum_{0}^{\infty} a_{rn-1} \, t^{-rn+1} ,$$

$$H = U\left(1 - \tfrac{1}{t^2}\right) - \tfrac{1}{2} i \omega \Big(t + \tfrac{1}{t^3} - n \sum_{1}^{\infty} K_m m t^{nm-1} \Big),$$

we may write

$$X - i Y = \tfrac{1}{2} i \rho \int_{C} \Big(G^2 \frac{dz}{dt} + H^2 \frac{dt}{dz} - 2 G H \Big) \, dt \tag{1.5.5}$$

and

$$M \doteq \text{real part of } -\tfrac{1}{2}\rho \int_C z\left(G^2 \frac{dz}{dt} + H^2 \frac{dt}{dz} - 2GH\right) dt \, .$$

(1.5.6)

Now

$$\int_C G^2 \frac{dz}{dt} \, dt = 0 \qquad (1.5.7)$$

and

$$\int_C GH \, dt = \pi\omega\left(U + \tfrac{1}{6} i\omega\right) \quad \text{when} \quad n = 3,$$

$$= \pi\omega U \quad \text{when} \quad n \geqslant 4 \, .$$

(1.5.8)

We have

$$\frac{dz}{dt} = \frac{1}{t^2}\left[-1 + \sum_{r=1}^{\infty} a_{rn-1}(rn-1)\, t^{rn}\right].$$

(1.5.9)

The series on the right of (1.5.9) is absolutely convergent for $|t| \leq 1$. Further

$$\left|\sum_1^{\infty} a_{rn-1}(rn-1)\, t^{rn}\right| < \sum_1^{\infty} \left|a_{rn-1}(rn-1)\right| .$$

$$< 1 .$$

Therefore

$$\frac{dt}{dz} = -t^2\left[1 + \sum_1^\infty a_{2n-1}(2n-1)\,t^{2n} + \right.$$

$$\left. + \left\{\sum_1^\infty a_{2n-1}(2n-1)\,t^{2n}\right\}^2 + \cdots\right].$$

So we find

$$\int_C H^2 \frac{dt}{dz}\,dt = -2\pi U\omega + \frac{\pi i}{3}\omega^2 \quad \text{when } n=3,$$

$$= -2\pi U\omega \qquad \text{when } n \geqslant 4.$$

(1.5.10)

Therefore for all values of $n \geqslant 3$

$$X - iY = -2\pi i \rho U\omega$$

or

$$X = 0\,,\quad Y = 2\pi\rho U\omega\,.$$

If we define the lift coefficient C_L as

$$C_L = \frac{Y}{\rho U\omega S},$$

where S is the area of cross-section of the cylinder, we find, using (1.3.7),

$$C_L = 2\pi(0.4106) \qquad \text{when } n = 3,$$

and

$$C_L = 2\pi (0 \cdot 3484) \qquad \text{When} \quad n = 4 .$$

Thus we find that C_L decreases as the number of sides of the polygon increases. For a circular cylinder $C_L = 2$.

The contribution to the real part from the first and third integrals in (1.5.6) is clearly zero. The contribution from the second integral is $2\pi U \omega$ when $n = 3$, and it is zero when $n \geq 4$. Therefore the moment is $-\pi \rho U \omega$ when $n = 3$, and it is zero when $n \geq 4$.

1.6. Cylinder of hypotrochoidal cross-section

The transformation

$$z = \mu \left(\frac{1}{t} + m t^n \right), \quad \mu > 0,$$

$$0 \leq m \leq \frac{1}{n}, \qquad (1.6.1)$$

where n is a positive integer, transforms the outside of the hypotrochoid in the z-plane on to the interior of the unit circle in the t-plane. The parametric representation of the curve in the z-plane corresponding to the circle $|t| = 1$ in the t-plane is given by

$$x = \mu (\cos \phi + m \cos n\phi),$$

$$y = \mu (- \sin \phi + m \sin n\phi) .$$

(1.6.2)

If $m = 0$ the curve is a circle; if $n = 1$ the curve is an ellipse. When $n = 1/m = 2$ and $n = 1/m = 3$, the corresponding curves have three and four cusps respectively and they resemble in shape a triangle and square respectively.

Proceeding in the same way as before, we get the stream function for the flow past the cylinder to be

$$\psi = \mu U\left(R - \frac{1}{R}\right)\sin\theta - \frac{1}{4}\omega\left[x^2 + y^2 + \right.$$

$$\left. + \mu^2\left\{\left(R^2 - \frac{1}{R^2}\right)\cos 2\theta - 2m R^{n+1}\cos(n+1)\theta\right\}\right]. \tag{1.6.3}$$

The thrust on the cylinder for $n \geqslant 2$ and $mn = 1$ may be calculated as before. We find

when $n = 2$

$$X = -\frac{1}{2}\pi\rho\mu\left(U^2 + \frac{1}{2}\mu^2\omega^2\right),$$

$$Y = \frac{3}{4}\pi\rho\mu^2 U\omega;$$

when $n = 3$

$$X = 0,$$

$$Y = \frac{1}{3}\pi\rho\mu^2 U\omega;$$

when $n = 4$

$$X = \frac{1}{8}\pi\rho\mu^3\omega^2,$$

$$Y = \frac{7}{8}\pi\rho\mu^2 U\omega;$$

when $n \geq 5$

$$X = 0 ,$$

$$Y = \left(1 - \frac{1}{2n}\right) \pi \rho \mu^2 U \omega .$$

The moment is found to be $-5/4\ \pi \rho \mu^3 U \omega$ when $n = 2$ and it is zero when $n \geq 3$.

Adjusting μ such that the arc length between two consecutive cusps of the hypotrochoid may correspond to the length of a side of the regular polygon, the results are tabulated below.

Cross-section	Drag	Lift	Moment
Equilateral triangle	0	$2\pi \rho U \omega$	$-\pi \rho U \omega$
Three-cusped hypotrochoid	$-(0.445\pi\rho U^2 + 0.176\pi\rho\omega^2)$	$0.593\pi\rho U\omega$	$-0.879 \times \pi\rho U\omega$
Square	0	$2\pi\rho U\omega$	0
Four-cusped hypotrochoid	0	$0.239\pi\rho U\omega$	0

Thus it appears that the lift is less in the case of cylinders with slightly curved edges as compared to cylinders having straight edges.

SHEET NO.

CHAPTER II

SLOW MOTION OF A PARABOLOID OF REVOLUTION IN A ROTATING FLUID+

2.1. Introduction

The motion of bodies in a rotating fluid has been a subject for a series of investigations in recent years. The perturbation caused by the motion of a body in an inviscid fluid exhibits different characteristics according as the fluid is at rest at infinity or is rotating about an axis there. Thus if the fluid is at rest at infinity, the flow is everywhere irrotational and dependent only on the instantaneous velocity of the body. But if the fluid is rotating about an axis, the perturbation in the fluid velocity depends not only on the instantaneous velocity of the body but also on its past history and is in general neither steady nor irrotational; and even in cases where a steady solution of the governing equation can be found, there is no guarantee that the flow can be set up by starting the body from rest relative to the rotating system. For these reasons it is necessary to consider an initial-value problem while dealing with this type of fluid motion.

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The exact steady-state equations of motion are non-linear. G.I.Taylor[26] has obtained a family of solutions of these equations for the case of steady motion of a sphere along the axis of rotation of the fluid. All the solutions satisfied the boundary condition at the surface of the sphere and also the condition that the relative velocity should vanish at infinity. It was thought possible that the indeterminacy lay in the manner in which the motion was started. Keeping this in view various attempts have been made to obtain some general features of the flow on the basis of the linearized equations.

When the body moves slowly it has been customary to use a small perturbation theory. Using this method Stewartson[23,24] has investigated the slow uniform motion, after an impulsive start from relative rest, of a sphere and an ellipsoid along the axis of a rotating liquid. In both these cases he found that ultimately the fluid inside the circumscribing cylinder $\mathcal{C}$ with its generators parallel to the axis of rotation is pushed along in front of the body as if it were solid, while outside the cylinder there is a shearing motion parallel to the axis of rotation. There is also a swirling motion about the axis inside the cylinder $\mathcal{C}$. The ultimate velocity distribution in the fluid is in general steady and two-dimensional (in the sense that the motion is the same in all planes perpendicular to the axis of rotation) everywhere except on the body and on its axis, where it oscillates finitely. In fact the linearized equations show that every slow and steady motion must also be two-dimensional.

There is, however, an 'a priori' difficulty with any theory which supposes that the perturbation remains small. Since the circulation round any material circular circuit concentric with the axis must remain constant, the radius of such a circuit must always be nearly equal to its initial radius. At first sight this restriction on the total strain of the fluid seems unlikely to be satisfied since it is inconsistent with any prolonged general streaming (however small) past the body. Nevertheless, Stewartson's[23,24] solutions do show that the circulation can remain constant and these solutions must therefore contain an explanation of the mechanism by which the circulation is maintained at a constant level, though Stewartson does not point this out. Moreover, his solution is very complicated, largely due to the formation of the singular surface at the cylinder $\mathscr{C}$, and this rather obscures the mechanism of the flow. In this chapter, therefore, a much simpler solution which does not have a surface corresponding to $\mathscr{C}$ is obtained, the primary aim being to illustrate the mechanism by which the circulation remains constant even when the perturbation remains small.

The flow considered here is that due to the slow uniform motion, started impulsively from relative rest, of a paraboloid of revolution along the axis of a rotating liquid. It is found that the radius of any material circuit concentric with the axis of rotation executes small oscillations which are 180° out of phase with the corresponding oscillations in the azimuthal velocity component. As time progresses, the amplitude of these oscillations decays to zero everywhere except on the paraboloid. The ultimate flow is then steady and two-dimensional everywhere except on the body and on the axis of rotation. On the body the velocity oscillates finitely and on the axis the velocity component parallel to the axis oscillates finitely and the other components are zero. The swirling motion about the axis found in the case of the sphere and ellipsoid is absent here.

2.2. Equations of motion

The origin O of a fixed set of cartesian axes (x', y', z') lies on the axis Oz' of steady rotation of the fluid. Let V' be the velocity vector of a point whose position vector is r. Then the equation of continuity and equation of motion of an inviscid incompressible fluid are

$$\nabla \cdot V' = 0 , \tag{2.2.1}$$

$$\frac{\partial V'}{\partial t} + V' \cdot \nabla V' = F - \frac{1}{\rho} \nabla p , \tag{2.2.2}$$

where ρ is the density of the fluid, p is the pressure and F is the body force per unit mass.

Let us now consider the motion relative to a rotating frame, which has the same origin as the fixed frame, but which has the constant angular velocity Ω about oz'. Let V be the velocity of an element of the fluid relative to this frame, so that

$$V' = V + \Omega \times r \quad . \tag{2.2.3}$$

Then equations (2.2.1) and (2.2.2) become[13]

$$\nabla \cdot V = 0 \; , \tag{2.2.4}$$

$$\frac{\partial V}{\partial t} + V \cdot \nabla V + 2\,\Omega \times V + \Omega \times (\Omega \times r) = F - \frac{1}{\rho} \nabla p \; . \tag{2.2.5}$$

The last two terms on the left hand side of (2.2.5) correspond to the Coriolis and centrifugal accelerations which are associated with the angular velocity Ω of the rotating frame.

To simplify the problem the equations of motion are linearized by assuming slow relative motion. If the equations are written in non-dimensional form, we find that $U/a\Omega = G$ is a dimensionless parameter of the problem. Here a is a characteristic length and U a characteristic velocity of the problem. Let us now assume that it is possible to write the solutions in the form of power series in the parameter G.

We see then that the coefficients of the first power must satisfy the following linear differential equations:

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial P}{\partial x}, \tag{2.2.6}$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -\frac{\partial P}{\partial y}, \tag{2.2.7}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}, \tag{2.2.8}$$

where

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2(x^2 + y^2) \tag{2.2.9}$$

and u, v, w are the components of the fluid velocity along x, y, z respectively. In writing down the above equations we have taken the body force $F = 0$. If F is not zero but is the gradient of a potential, the analysis is unaffected provided that the pressure p is taken to be the difference between the actual pressure and the pressure when the fluid is at rest.

It must be pointed out, however, that the conditions under which it is justifiable to neglect the non-linear terms in the equations of motion are not yet clearly established. It is not always possible to decide beforehand whether the velocities associated with a particular type of motion will or will not be small. For example, the motion ahead of a sphere which is moving slowly along the axis of rotation is of the required type while the motion behind is not. This is indicated by the experiments made by G.I.Taylor[27] and R.R.Long[10]. Taylor found that if $U/a\Omega$ was less than about 0.3, where a is the radius of the sphere and U the uniform velocity of the sphere along the axis, the fluid within $\mathcal{C}$ and ahead of the sphere was pushed along by it. Long's experiments were made with a body having a spherical nose and a conical tail so as to minimize the effect of boundary layer separation. He found that for $U/a\Omega$ less than about 0.2, the fluid in the central part of $\mathcal{C}$ was pushed ahead of the body, but that the fluid near the boundary passed round to the rear of the body; the fluid to the rear of the body and inside $\mathcal{C}$ was not carried along with it.

The solution for a sphere obtained by Stewartson[23] on the basis of the linearized equations agrees well with Taylor's experimental observations[27]. But the solution fails to explain Long's[10] observations on the flow to the rear of the body. One possible reason for this failure perhaps lies in the fact that, as mentioned above, the flow behind the body is not of the type which allows linearization. However, it must be remembered that the experiments[27,10] were carried out with fluid inside a tube of finite radius ℓ with a/ℓ small. It may be therefore argued that the character of the flow completely changes as a/l tends to zero and probably this is the reason for the discrepancy between theory and experiment. On the whole, Stewartson's solution[23] may be said to be in agreement with the general features of the experiments. Stewartson has also pointed out that the modifications caused by retaining the non-linear terms in the full equations of motion are likely to be confined to the neighbourhood of the singular surface $\mathcal{C}$. It therefore appears reasonable to expect the linearized equations to bring out at least some of the general features of the actual state of affairs.

2.3. Solution to the problem

We choose cylindrical polar coordinates, oz along the axis of rotation and (r, θ) polar coordinates in a plane normal to oz. Let the unperturbed motion of the fluid consist of a uniform angular velocity Ω about the z-axis. A paraboloid of revolution (whose axis of symmetry coincides with the z-axis) impulsively starts to move along the axis at $t = 0$ with uniform velocity V. If we choose the origin of coordinates to be in the body, we have in effect superposed a uniform velocity $-V$ on the system and brought the body to rest. Let the components of the fluid velocity along the directions of increasing r, θ, z be u, $\Omega r + v$, w respectively, where u, v, w are small. Then the linearized equations of motion are

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial P}{\partial r},$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0,$$

(2.3.1)

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \qquad (2.3.2)$$

where

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2 r^2 .$$

The boundary conditions are that

$$u \longrightarrow 0, \quad v \longrightarrow 0, \quad w \longrightarrow -V$$

as $z \longrightarrow \infty$ for fixed r, t,

(2.3.3)

and, on the body, the component of the fluid velocity normal to the body is zero.

As Morgan(14) pointed out, the initial disturbance travels with infinite velocity and the initial motion relative to the rotating system must be the irrotational motion with the given boundary conditions. Taking the velocity potential of this irrotational flow to be

$$\phi(r,z) = -Vz + \chi(r,z),$$

we have at $t = 0$

$$u = \frac{\partial \chi}{\partial r}, \quad v = 0, \quad w = -V + \frac{\partial \chi}{\partial z}.$$

Now to take the Laplace transforms of u, v, w and P, we put

$$\bar{u} = \int_0^\infty \exp(-st)\, u(r,z,t)\, dt, \quad etc.$$

Then (2.3.1) and (2.3.2) become

$$s\bar{u} - 2\Omega\bar{v} = -\frac{\partial \bar{N}}{\partial r},$$

$$s\bar{v} + 2\Omega\bar{u} = 0,$$

$$s\bar{w} + V = -\frac{\partial \bar{N}}{\partial z}, \qquad (2.3.4)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0, \qquad (2.3.5)$$

where $\bar{N} = \bar{P} - \chi$,

and the boundary conditions (2.3.3) become

$$\bar{u} \longrightarrow 0\,, \quad \bar{v} \longrightarrow 0\,, \quad \bar{w} \longrightarrow -\frac{V}{s}$$

$$\text{as} \quad z \longrightarrow \infty\,. \tag{2.3.6}$$

From (2.3.4) we get

$$\bar{u} = -\frac{s}{s^2 + 4\Omega^2}\frac{\partial \bar{N}}{\partial r}\,,$$

$$\bar{v} = \frac{2\Omega}{s^2 + 4\Omega^2}\frac{\partial \bar{N}}{\partial r}\,,$$

$$\bar{w} = -\frac{V}{s} - \frac{1}{s}\frac{\partial \bar{N}}{\partial z}\,, \tag{2.3.7}$$

so that the continuity condition (2.3.5) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{N}}{\partial r}\right) + \frac{s^2 + 4\Omega^2}{s^2}\frac{\partial^2 \bar{N}}{\partial z^2} = 0\,, \tag{2.3.8}$$

and the boundary condition (2.3.6) becomes

$$\frac{\partial \bar{N}}{\partial z} \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty\,. \tag{2.3.9}$$

If the section of the paraboloid in the (z, r)-plane is $z = -a r^2$, the condition on the body is

$$2 a r u + w = 0\,,$$

or

$$2ar\bar{u} + \bar{\omega} = 0 ,$$

or, in view of (2.3.7),

$$2ar \frac{s^2}{s^2 + 4\Omega^2} \frac{\partial \bar{N}}{\partial r} + \frac{\partial \bar{N}}{\partial z} = -V . \qquad (2.3.10)$$

So equation (2.3.8) is to be solved with boundary conditions (2.3.9) and (2.3.10).

Now we can easily formulate the problem in a coordinate system in which (2.3.8) can be solved simply and in which the body is a coordinate surface, by taking a suitable transformation of independent variables. We introduce new coordinates ξ , η defined by

$$z + \frac{k^2}{4a} = k (\xi^2 - \eta^2) ,$$

$$r = 2 \xi \eta , \qquad (2.3.11)$$

where

$$k^2 = \frac{s^2 + 4\Omega^2}{s^2} .$$

On the paraboloid we have

$$\xi = \xi_0 = \left(\frac{k}{4a}\right)^{1/2} .$$

With the above transformation equation (2.3.8) becomes

$$\frac{\partial^2 \bar{N}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \bar{N}}{\partial \xi} + \frac{\partial^2 \bar{N}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \bar{N}}{\partial \eta} = 0 \, ; \tag{2.3.12}$$

and the boundary condition (2.3.10) becomes

$$\frac{\partial \bar{N}}{\partial \xi} = -2KV\xi_0 \quad \text{on} \quad \xi = \xi_0 \, . \tag{2.3.13}$$

The appropriate solution of (2.3.12) is

$$\bar{N} = (A + B \log \xi)(C + D \log \eta) \, ,$$

and using (2.3.13) we get

$$\bar{N} = -2KV\xi_0 \log \xi \, . \tag{2.3.14}$$

Now

$$\frac{\partial \bar{N}}{\partial z} = \frac{1}{2K(\xi^2+\eta^2)}\left(\xi \frac{\partial \bar{N}}{\partial \xi} - \eta \frac{\partial \bar{N}}{\partial \eta}\right)$$

$$= -\frac{V\xi_0^2}{\xi^2+\eta^2} \, .$$

Therefore

$$\frac{\partial \bar{N}}{\partial z} \longrightarrow 0 \quad \text{as} \quad \xi \longrightarrow \infty \, ,$$

in agreement with (2.3.9). Thus (2.3.14) is the appropriate solution.

The results for $\bar{u}$, $\bar{v}$, $\bar{w}$ follow immediately from (2.3.7). Finally, inverting these Laplace transforms, we find that the velocity components at any point of the fluid are given by

$$u = \frac{V}{8ak\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[1 - \frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right]\frac{e^{st}}{s}\,ds , \tag{2.3.15}$$

$$v = -\frac{\Omega V}{4ak\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[1 - \frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right]\frac{e^{st}}{s^2}\,ds , \tag{2.3.16}$$

$$w = -V + \frac{V}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^2+4\Omega^2}{\omega^2}\,e^{st}\,ds , \tag{2.3.17}$$

where $\gamma > 0$ and

$$\omega^2 = \left\{(1+4az)^2+16a^2k^2\right\}^{1/2}\left(s^2+4\Omega^2 l_1^2\right)^{1/2}\left(s^2+4\Omega^2 l_2^2\right)^{1/2},$$

$$l_1^2 = \frac{1 + 4az + 8a^2r^2 - 8ar(a^2r^2 + az)^{1/2}}{(1+4az)^2 + 16a^2r^2},$$

$$l_2^2 = \frac{1 + 4az + 8a^2r^2 + 8ar(a^2r^2 + az)^{1/2}}{(1+4az)^2 + 16a^2r^2}.$$

These results represent a complete formal solution to the problem. For the present purpose of ascertaining the general features of the flow, however, it is only necessary to consider certain special cases of the formulae (2.3.15), (2.3.16) and (2.3.17).

2.4. General features of the flow

On the surface of the paraboloid the integrals (2.3.15), (2.3.16), (2.3.17) simplify considerably and it is possible to evaluate them in the forms

$$u = \frac{2Var}{1+4a^2r^2} \cos \frac{2\Omega t}{(1+4a^2r^2)^{1/2}},$$

$$v = -\frac{2Var}{(1+4a^2r^2)^{1/2}} \sin \frac{2\Omega t}{(1+4a^2r^2)^{1/2}},$$

$$w = -\frac{4Va^2r^2}{1+4a^2r^2} \cos \frac{2\Omega t}{(1+4a^2r^2)^{1/2}} \tag{2.4.1}$$

Thus we find that the motion never becomes steady on the paraboloid. More important, perhaps, these results show very simply the way in which the circulation round circular material circuits, concentric with the axis of rotation and lying on the paraboloid, remains constant. Since the radial velocity oscillates sinusoidally in time, the radius of such a circuit must oscillate in a similar way, and the primary rotation then makes an oscillatory contribution to the circulation round the circuit. This must, in turn, be counterbalanced by an oscillatory contribution from the azimuthal perturbation velocity v, in accordance with (2.4.1). As far as the validity of the linearized analysis is concerned, the essential feature of this mechanism is that no fluid particle is displaced appreciably in a radial direction from its initial position.

Similarly, on the axis of rotation we find

$$u = 0 ,$$

$$v = 0 ,$$

$$w = -\frac{4Vaz}{1+4az}\cos\frac{2\Omega t}{(1+4az)^{1/2}} \qquad (2.4.2)$$

Here again the oscillatory axial velocity implies that small material circuits surrounding the axis of rotation are never swept on to the surface of the paraboloid, thereby increasing their perimeter by a large factor, an essential result if the azimuthal perturbation velocity is to remain small.

From these special cases it is reasonable to infer that the same oscillatory mechanism is responsible for maintaining the radial positions of all fluid particles, and this may be verified directly when the motion is approaching its ultimate form. Thus, for large values of t, the integrals in (2.3.15), (2.3.16), (2.3.17) may be evaluated by inserting cuts in the s-plane from $s = \pm 2i\Omega \ell_1$, and $s = \pm 2i\Omega \ell_2$ along lines on which the imaginary part of s is constant and the real part decreases. The path of integration may now be replaced by a path round the infinite semicircle $\mathcal{R}\{s\} < 0$ and round the four cuts. For example, the contribution from the branch point $s = 2i\Omega \ell_1$ to the integral in (2.3.15) is found to be

$$e^{2i\Omega \ell_1 t} \frac{1-(1+4az)\ell_1^2}{\ell_1(\ell_2^2-\ell_1^2)^{1/2}(4i\Omega \ell_1)^{1/2}} \frac{\Gamma(\frac{1}{2})}{t^{1/2}}$$

for large t. In this way we find that

$$u \sim -\frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)\ell_1^2}{\ell_1(\Omega \ell_1)^{1/2}} \sin(2\Omega \ell_1 t - \tfrac{\pi}{4}) + \right.$$

$$\left. + \frac{1-(1+4az)\ell_2^2}{\ell_2(\Omega \ell_2)^{1/2}} \sin(2\Omega \ell_2 t + \tfrac{\pi}{4})\right] \frac{1}{(\pi t)^{1/2}},$$

(2.4.3)

$$v \sim \frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)l_1^2}{l_1^2(\Omega l_1)^{1/2}}\cos\left(2\Omega l_1 t - \frac{\pi}{4}\right) + \right.$$

$$\left. + \frac{1-(1+4az)l_2^2}{l_2^2(\Omega l_2)^{1/2}}\cos\left(2\Omega l_2 t + \frac{\pi}{4}\right)\right]\frac{1}{(\pi t)^{1/2}}, \tag{2.4.4}$$

$$w \sim \frac{V}{4(ar)^{1/2}(a^2r^2+az)^{1/4}}\left[\frac{1-l_1^2}{l_1(\Omega l_1)^{1/2}}\sin\left(2\Omega l_1 t - \frac{\pi}{4}\right) + \right.$$

$$\left. + \frac{1-l_2^2}{l_2(\Omega l_2)^{1/2}}\sin\left(2\Omega l_2 t + \frac{\pi}{4}\right)\right]\frac{1}{(\pi t)^{1/2}}. \tag{2.4.5}$$

Thus the only significant difference here is that the amplitude of the oscillations decreases to zero, so that the ultimate motion is in general steady and two-dimensional and the axial velocity of the fluid is ultimately the same as that of the paraboloid.

In view of the ultimate singularity in the velocity gradients on the axis and body it seems that the detailed form (but probably not the general nature) of the solution is of doubtful validity in this neighbourhood.

SHEET NO.

CHAPTER III

ON THE FLOW OF A ROTATING FLUID PAST A SPHERE IN A CYLINDRICAL PIPE

3.1. Introduction

A number of investigations regarding the slow motion of bodies in an inviscid rotating liquid of infinite extent have been carried out by Proudman[17], Grace[6], Görtler[5], Morgan[13] and Stewartson[23,24]. All these investigations are based on the linearized equations (2.3.1) and some interesting results in agreement with experiments are obtained. The solutions of the exact steady state equations for a sphere obtained by Taylor[26] and Long[9], though incomplete, indicate the wave character of the motion. Long[9] showed that when a symmetrical obstacle moves steadily along the axis of a rotating liquid, the general equations of motion reduce to a single linear differential equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + K^2 \psi = -\frac{K^2}{2} U r^2, \tag{3.1.1}$$

where z and r are the cylindrical coordinates of any point in the meridian plane and $K = 2\Omega/U$, U being the uniform velocity of the body along the z-axis and Ω the uniform angular velocity of the liquid about the z-axis; if u, v, w are the velocity components of the fluid in the directions of r, ϕ and z respectively, then they are given by the relations

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z},$$

$$v = -\frac{K\psi}{r},$$

$$w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \qquad (3.1.2)$$

Expressing (3.1.1) in spherical polar coordinates, Long obtained the solution for the sphere as

$$\psi = -\frac{U}{2} R^2 \sin^2\theta +$$

$$+ (KR)^{1/2} \sin\theta \sum_{n=1}^{\infty} \left[A_n J_{n+\frac{1}{2}}(KR) + B_n J_{-n-\frac{1}{2}}(KR) \right] P_n^{(1)}(\cos\theta), \qquad (3.1.3)$$

where $J_{n+\frac{1}{2}}$ is a Bessel function, $P_n(\cos\theta)$ is a Legendre polynomial and A_n, B_n are constants. This expression satisfies the condition of uniform flow at infinity and uniform rotation about the axis. The condition of zero normal velocity on the body provides only one relation to determine the constants A_n and B_n. Thus the problem is left indeterminate. The solution given in (3.1.3) indicates the wave character of motion. Ofcourse it is to be expected that a rotating liquid would generate waves when a symmetrical obstacle moves uniformly along its axis. This important feature is not, however, brought out in the analysis based on the linearized equations. It is mostly due to the unboundedness of the fluid and the neglect of the inertia terms in the equations of motion. Moreover, the linearized theory is based on the assumption that the dimensionless parameter $2a\Omega/U$ is sufficiently large. The ultimate flow in this case is of a cylindrical character; there is no radial velocity and the flow is the same in every plane normal to the axis of rotation. Squire[22] has pointed out that the exact differential equation of steady flows, equation (3.1.1) above, suggests two main possibilities for large values of $2a\Omega/U = ka$. The first possibility is that the azimuthal component of vorticity increases with ka, no matter how large the value of ka and the motion becomes oscillatory in character. The second possibility is that the azimuthal component of vorticity remains bounded when ka becomes sufficiently large; in this case equation (3.1.1) shows that the stream function ψ is approximately a function of the radial distance from the axis only, so that the flow is cylindrical. Taylor's experiments[27] indicate that the second alternative is the one found in practice for sufficiently large ka. In view of the results obtained in Chapter II of this dissertation and the results obtained in the present investigation below, we conclude that for large ka the motion is oscillatory in character, which gives way to the cylindrical flow ultimately (that is, as $t \longrightarrow \infty$). If the inertia terms are partly taken into account (in the manner of Oseen) the analysis based on the linearized equations[22] may be expected to throw further light on this question.

Squire[22] pointed out that the indeterminacy in the solution for a sphere (3.1.3) may be removed by taking into account the presence of a cylindrical boundary at a finite distance. Thus if the fluid is contained in a cylindrical vessel of radius l, for small values of k the solution of (3.1.1) may be taken as

$$\psi = -\tfrac{1}{2} U r^2 + r \sum_{n=1}^{\infty} A_n \exp\left(\mp l_n \frac{z}{l}\right) J_1\left(j_n \frac{r}{l}\right), \tag{3.1.4}$$

where J_1 is the Bessel function of the first order, j_n is the n th zero of this function, so that $J_1(j_n) = 0$, and $l_n^2 = j_n^2 - k^2 l^2$. The negative and positive signs are to be taken for the regions $z > 0$ and $z < 0$ respectively. The first three values of j_n are $j_1 = 3 \cdot 83$, $j_2 = 7 \cdot 02$, $j_3 = 10 \cdot 17$. All the values of l_n are therefore real if $kl < 3 \cdot 83$. It may be therefore concluded that for $\kappa l < 3 \cdot 83$, the swirling flow past a body, such as a sphere, will resemble in its general features the flow without swirl, and that the presence of a boundary at a finite distance will, in these circumstances, remove the indeterminacy which arises for a body in an unbounded fluid.

On the other hand, if $j_{m+1} > \kappa l > j_m$, the solution of (3.1.1) which is symmetrical with respect to $z = 0$ is of the form

$$\psi = -\tfrac{1}{2} U r^2 + r \sum_{n=1}^{m} \left[A_n \cos\left(l_n \frac{z}{l}\right) \mp B_n \sin\left(l_n \frac{z}{l}\right) \right] J_1\left(j_n \frac{r}{l}\right)$$

$$+ r \sum_{n=m+1}^{\infty} A_n \exp\left(\mp l_n \frac{z}{l}\right) J_1\left(j_n \frac{r}{l}\right), \tag{3.1.5}$$

where

$$l_n^2 = \kappa^2 l^2 - j_n^2 \qquad \text{for} \quad n < m,$$

and

$$l_n^2 = j_n^2 - \kappa^2 l^2 \qquad \text{for} \quad n > m,$$

the B_n are further arbitrary constants, and, as before, the upper signs are taken for $z > 0$, and the lower signs for $z < 0$. This form of the solution shows that wave motion may arise. The wave-length in the stream direction of the shortest wave is

$$\lambda_1 = 2\pi\left[k^2 - \left(\frac{j_1}{l}\right)^2\right]^{-1/2} . \tag{3.1.6}$$

Equation (3.1.5) may be taken to determine a solution symmetrical about the origin (that is, even in z), which will include waves both upstream and downstream. But the experiments of R.R.Long[10] showed that waves are propagated downstream only. Now assuming the body to be small compared to the diameter of the boundary cylinder, we can subtract the quantity

$$r \sum_{n=1}^{m} \left[A_n \cos\left(l_n \frac{z}{l}\right) + B_n \sin\left(l_n \frac{z}{l}\right)\right] J_1\left(j_n \frac{r}{l}\right) \tag{3.1.7}$$

from the expression (3.1.5) for ψ , without affecting the flow near the body to the first order. The above quantity represents a possible standing wave motion in the unobstructed tube. This procedure enables the wave-like part of the solution (3.1.5) to be removed far upstream and corresponds to the procedure devised by Raleigh to make some surface-wave problems determinate[8]. Raleigh showed that a small viscosity was sufficient to damp out any upstream surface-waves in the problems he considered and that such problems remain determinate, even if the viscosity tends to zero.

By using the above procedure, Fraenkel[3] investigated the flow of a rotating liquid past small bodies in a pipe on the basis of the exact steady-state equations. Though the problem of the flow past a body of finite size in a cylindrical tube has not yet been resolved, the above investigation brought out some general features of the flow which may be expected not to differ essentially from those for finite bodies. The solutions obtained by Fraenkel[3], however, apply only for a limited range of $K\ell$, where ℓ is the radius of the tube. The upper limit of this range, where flow with no disturbances far upstream (as found in Long's experiments) begins to give way to cylindrical flow (as found in Taylor's experiments), is as yet unknown. However, the character of the flow for larger values of $K\ell$ can be ascertained by using the linearized equations. Moreover, before proceeding to make the solution (3.1.3) unique by annuling the upstream waves, it appears necessary to provide some theoretical justification for the absence of upstream oscillations on the basis of further analysis.

The present investigation studies the flow of rotating liquid past a sphere in a cylindrical pipe on the basis of the linearized equations. Assuming the pipe radius to be large compared to the radius of the sphere, it is found that the flow is oscillatory in character. Further, it appears that the disturbances are not likely to appear far upstream and waves are propagated downstream only. The ultimate flow is steady and two-dimensional.

3.2. Solution to the problem

We choose cylindrical polar coordinates oz along the axis of rotation and r, ϕ polar coordinates in a plane normal to oz. A sphere of radius a impulsively starts to move with uniform velocity V along the axis oz of a cylindrical pipe of radius b, filled with liquid rotating with uniform angular velocity Ω about the axis. On taking Laplace transforms with respect to time of u, v, w, the perturbation velocities of the fluid along the directions of increasing r, ϕ, z, the equations of motion become (cf. 2.3.4)

$$s\bar{u} - 2\Omega\bar{v} = -\frac{\partial \bar{N}}{\partial r}, \tag{3.2.1}$$

$$s\bar{v} + 2\Omega\bar{u} = 0, \tag{3.2.2}$$

$$s\bar{w} + V = -\frac{\partial \bar{N}}{\partial z}, \tag{3.2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0. \tag{3.2.4}$$

The boundary conditions are

(i) $\bar{u} \longrightarrow 0\,,\ \bar{v} \longrightarrow 0\,,\ \bar{w} \longrightarrow -\dfrac{V}{s}$

as $z \longrightarrow \infty$, (3.2.5)

(ii) $\bar{u} = 0$ on $r = l$ (3.2.6)

and (iii) normal velocity is zero on the sphere . (3.2.7)

Putting

$$\bar{u} = \frac{1}{r}\frac{\partial \psi}{\partial z}\,, \quad \bar{w} = -\frac{1}{r}\frac{\partial \psi}{\partial r}\,,$$

equation (3.2.4) is satisfied and elimination of $\bar{N}$ between (3.2.1) and (3.2.3) gives

$$\frac{s^2 + 4\Omega^2}{s^2}\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0\,. \tag{3.2.8}$$

The solution of (3.2.8) consistent with the boundary conditions (3.2.5) and (3.2.6) is

$$\psi = \frac{V}{2s} r^2 + r\sum_{i=1}^{\infty} A_i \exp\left(\mp \frac{k_i}{l} z\right) J_1\left(\frac{\alpha_i}{l} r\right), \tag{3.2.9}$$

where

$$K_i = \alpha_i \frac{s}{(s^2 + 4\Omega^2)^{1/2}}$$

and α_i are the zeros of $J_1(\alpha)$. The negative and positive signs in the exponent are to be taken for the regions $z > 0$ and $z < 0$ respectively. In order to satisfy the boundary condition on the sphere, we write (3.2.9) in terms of R, θ, where $R \cos\theta = z$ and $R \sin\theta = r$.

We have[31]

$$J_1\left(\frac{\alpha_i}{l} R \sin\theta\right) = \left(\frac{1}{2}\pi \frac{\alpha_i}{l} R\right)^{-1/2} \Gamma\left(\tfrac{3}{2}\right) \sin\theta \sum_{m=0}^{\infty} \frac{(2m+\frac{3}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} \times$$

$$\times\, C_{2m}^{3/2}(\cos\theta)\, J_{2m+\frac{3}{2}}\left(\frac{\alpha_i}{l} R\right),$$

where $C_{2m}^{3/2}(\cos\theta)$ is Gegenbauer polynomial.

Also we have[31]

$$C_{2m}^{3/2}(\cos\theta) = P_{2m+1}'(\cos\theta),$$

where $P_{2m+1}(\cos\theta)$ is Legendre polynomial and dash denotes differentiation with respect to $\cos\theta$. Now using the result[28]

$$\exp\left(\frac{k_i}{\ell} R\cos\theta\right) P'_{2m+1}(\cos\theta)$$

$$= \sum_{n=1}^{\infty} g_{2m+1,n}\left(\frac{k_i}{\ell}R\right) P'_n(\cos\theta),$$

where

$$g_{2m+1,n}(\xi) = \frac{4(2n+1)}{n(n+1)} \sum_{j=0}^{2m} \frac{(2j+1)!(4m+1-2j)!(2n-2j-1)!}{(j!)^2(4m+2+2n-2j)!} \times$$

$$\times \left\{\frac{(2m+1+n-j)!}{(2m-j)!(n-j-1)!}\right\}^2 \frac{X_{2m+n-2j}(\xi)}{\xi},$$

$$X_n(\xi) = (2n+1)\left(\frac{\pi}{2\xi}\right)^{1/2} I_{n+\frac{1}{2}}(\xi),$$

$I_{n+\frac{1}{2}}(\xi)$ being a modified Bessel function, we get

$$\psi = \frac{V}{2s} R^2 \sin^2\theta +$$

$$+ R\sin^2\theta \sum_{i=1}^{\infty} A_i\left(\frac{2d_i}{\ell}R\right)^{-1/2} \sum_{m=0}^{\infty} \frac{(2m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} \times$$

$$\times J_{2m+\frac{3}{2}}\left(\frac{d_i}{\ell}R\right) \sum_{n=1}^{\infty} g_{2m+1,n}\left(\frac{k_i}{\ell}R\right) P'_n(\cos\theta). \qquad (3.2.10)$$

Now using the condition (3.2.7) on the sphere, viz. $\partial\psi/\partial\theta = 0$ when $R = a$, and making use of the Legendre equation

$$\sin^2\theta\, P_n''(\cos\theta) - 2\cos\theta\, P_n'(\cos\theta) + n(n+1)\, P_n(\cos\theta) = 0,$$

we get

$$\frac{Va}{s} + \sum_{i=1}^{\infty} A_i \left(\alpha_i \frac{a}{\ell}\right)^{-1/2} \sum_{m=0}^{\infty} \frac{(2m+\frac{3}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} \times$$

$$\times J_{2m+\frac{3}{2}}\left(\alpha_i \frac{a}{\ell}\right) g_{2m+1,1}\left(k_i \frac{a}{\ell}\right) = 0 \qquad (3.2.11)$$

and

$$\sum_{i=1}^{\infty} A_i \left(\alpha_i \frac{a}{\ell}\right)^{-1/2} \sum_{m=0}^{\infty} \frac{(2m+\frac{3}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} J_{2m+\frac{3}{2}}\left(\alpha_i \frac{a}{\ell}\right) \times$$

$$\times g_{2m+1,n}\left(k_i \frac{a}{\ell}\right) = 0,$$

$$n = 2, 3, \ldots\ldots \qquad (3.2.12)$$

The A_i can be now determined by solving these simultaneous equations.

3.3. General features of the flow

In order to ascertain some of the general features of the flow, we assume that the radius of the cylinder is large compared to the radius of the sphere; now retaining the terms corresponding to $m = 0$ only in (3.2.11) and (3.2.12) we get

$$\frac{Va}{\delta} = \sum_{i=1}^{\infty} A_i \left(\alpha_i \frac{a}{l}\right)^{-1/2} J_{3/2}\left(\alpha_i \frac{a}{l}\right)\left(\frac{\pi}{2 K_i \frac{a}{l}}\right)^{1/2} \frac{I_{3/2}\left(K_i \frac{a}{l}\right)}{K_i \frac{a}{l}}.$$

(3.3.1)

and

$$\sum_{i=1}^{\infty} A_i \left(\alpha_i \frac{a}{l}\right)^{-1/2} J_{3/2}\left(\alpha_i \frac{a}{l}\right)\left(\frac{\pi}{2 K_i \frac{a}{l}}\right)^{1/2} \frac{I_{5/2}\left(K_i \frac{a}{l}\right)}{K_i \frac{a}{l}} = 0.$$

(3.3.2)

By carrying out the detailed calculation it is found that the values of the A_i decrease fairly rapidly as i increases. Therefore as the first approximation we may take for $z > 0$

$$\psi = \frac{V}{2\delta} r^2 + \frac{Va}{\delta}\frac{r}{\Delta}\Big[I_{5/2}\left(K_2 \frac{a}{l}\right) \exp\left(-K_1 \frac{z}{l}\right) J_1\left(\alpha_1 \frac{r}{l}\right) - $$

$$- I_{5/2}\left(K_1 \frac{a}{l}\right) \exp\left(-K_2 \frac{z}{l}\right) J_1\left(\alpha_2 \frac{r}{l}\right)\Big],$$

(3.3.3)

where

$$\Delta = I_{5/2}\left(K_2 \frac{a}{l}\right) I_{3/2}\left(K_1 \frac{a}{l}\right) - I_{3/2}\left(K_2 \frac{a}{l}\right) I_{5/2}\left(K_1 \frac{a}{l}\right).$$

Now u, v, w can be written down in the forms

$$u = V\frac{a}{l}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\Delta}\Big[J_1\Big(\frac{\alpha_1}{l}r\Big)K_1 I_{5/2}\Big(K_2\frac{a}{l}\Big)\exp\Big(-\frac{K_1}{l}z\Big) -$$

$$-J_1\Big(\frac{\alpha_2}{l}r\Big)K_2 I_{5/2}\Big(K_1\frac{a}{l}\Big)\exp\Big(-\frac{K_2}{l}z\Big)\Big]\frac{e^{st}}{s}\,ds\,,$$

(3.3.4)

$$v = -2\Omega V\frac{a}{l}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\Delta}\Big[J_1\Big(\frac{\alpha_1}{l}r\Big)K_1 I_{5/2}\Big(K_2\frac{a}{l}\Big)\exp\Big(-\frac{K_1}{l}z\Big) -$$

$$-J_1\Big(\frac{\alpha_2}{l}r\Big)K_2 I_{5/2}\Big(K_1\frac{a}{l}\Big)\exp\Big(-\frac{K_2}{l}z\Big)\Big]\times$$

$$\times\frac{e^{st}}{s^2}\,ds\,,$$

(3.3.5)

$$w = -V - V\frac{a}{l}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\Delta}\Big[\frac{\alpha_1}{l}J_0\Big(\frac{\alpha_1}{l}r\Big)I_{5/2}\Big(K_2\frac{a}{l}\Big)\exp\Big(-\frac{K_1}{l}z\Big) -$$

$$-\frac{\alpha_2}{l}J_0\Big(\frac{\alpha_2}{l}r\Big)I_{5/2}\Big(K_1\frac{a}{l}\Big)\exp\Big(-\frac{K_2}{l}z\Big)\Big]\frac{e^{st}}{s}\,ds\,.$$

(3.3.6)

When Ωt is small the values of u, v, w can be found by expanding the integrand in a series of descending powers of s. Thus we get

$$u = \frac{3}{2} V \left(\frac{\pi}{2} \frac{a}{l} \right)^{1/2} \left(\frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} \right) \times$$

$$\times \left[(\alpha_1)^{-3/2} \exp\left(-\frac{\alpha_1}{l} z\right) \left\{ 1 + \left(2 \frac{\alpha_1}{l} z + 1\right) \frac{\Omega^2 t^2}{2!} + \cdots \right\} J_1\left(\frac{\alpha_1}{l} r\right) - \right.$$

$$\left. - (\alpha_2)^{-3/2} \exp\left(-\frac{\alpha_2}{l} z\right) \left\{ 1 + \left(2 \frac{\alpha_2}{l} z + 1\right) \frac{\Omega^2 t^2}{2!} + \cdots \right\} J_1\left(\frac{\alpha_2}{l} r\right) \right], \tag{3.3.7}$$

$$v = -3 \Omega V t \left(\frac{\pi}{2} \frac{a}{l} \right)^{1/2} \left(\frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} \right) \times$$

$$\times \left[(\alpha_1)^{-3/2} \exp\left(-\frac{\alpha_1}{l} z\right) \left\{ 1 + \left(\frac{2\alpha_1}{l} z + 1\right) \frac{\Omega^2 t^2}{3!} + \cdots \right\} J_1\left(\frac{\alpha_1}{l} r\right) - \right.$$

$$\left. - (\alpha_2)^{-3/2} \exp\left(-\frac{\alpha_2}{l} z\right) \left\{ 1 + \left(\frac{2\alpha_2}{l} z + 1\right) \frac{\Omega^2 t^2}{3!} + \cdots \right\} J_1\left(\frac{\alpha_2}{l} r\right) \right], \tag{3.3.8}$$

$$w = -V + 3V\left(\frac{\pi}{2}\frac{a}{l}\right)^{1/2}\left(\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\right)\frac{1}{l}\times$$

$$\times\Big[(\alpha_1)^{-3/2}\exp\left(-\frac{\alpha_1}{l}z\right)\left\{1+\left(\frac{2\alpha_1}{l}z+3\right)\Omega^2t^2+\cdots\right\}J_0\left(\frac{\alpha_1}{l}r\right)-$$

$$-(\alpha_2)^{-3/2}\exp\left(-\frac{\alpha_2}{l}z\right)\left\{1+\left(\frac{2\alpha_2}{l}z+3\right)\Omega^2t^2+\cdots\right\}J_0\left(\frac{\alpha_2}{l}r\right)\Big].$$

(3.3.9)

The above results show that for small Ωt the perturbation velocity components u, v, w remain small. When Ωt is large the integrals in (3.3.4), (3.3.5) and (3.3.6) may be evaluated by choosing a suitable contour. In view of the approximation we have made, the integrands have branch points at $s = 0$ and $s = \pm 2i\Omega$. We insert cuts in the s-plane from $s = 0$ and $s = \pm 2i\Omega$ to infinity along lines on which the imaginary part of s is constant and the real part decreases. The path of integration may be now replaced by a path round the infinite semicircle $\mathcal{R}e\{s\} < 0$ and round the three cuts provided that $t > \alpha_2 z/2\Omega l$. The contribution to the first term in the integral in (3.3.4) from the branch point $s = 0$ is found to be

$$V \frac{a}{l} \frac{1}{\pi} J_1\left(\frac{\alpha_1}{l} r\right) \int_0^\infty (\alpha^2 + 4\Omega^2)^{1/4} \exp\left\{ \frac{\alpha_1}{l} z \frac{\alpha}{(\alpha^2 + 4\Omega^2)^{1/2}} \right\} \times$$

$$\times\, e^{-\alpha t}\, d\alpha .$$

(3.3.10)

The contributions from the branch points $\delta = \pm 2i\Omega$ are respectively

$$V \frac{a}{l} \frac{1}{2\pi i} J_1\left(\frac{\alpha_1}{l} r\right) e^{2i\Omega t} \int_0^\infty \left[\exp\left\{ \frac{-\alpha_1 z (2i\Omega - \alpha)}{(4i\Omega\alpha - \alpha^2)^{1/2}} + \frac{\pi}{4} \right\} i \, - \right.$$

$$\left. - \exp\left\{ \frac{\alpha_1 z (2i\Omega - \alpha)}{(4i\Omega\alpha - \alpha^2)^{1/2}} + \frac{\pi}{4} \right\}(-i) \right] \times$$

$$\times \frac{(4i\Omega\alpha - \alpha^2)^{1/4}}{(2i\Omega - \alpha)^{3/2}}\, e^{-\alpha t}\, d\alpha$$

(3.3.11)

and

$$V \frac{a}{l} \frac{1}{2\pi i} J_1\left(\frac{\alpha_1}{l} r\right) e^{-2i\Omega t} \int_0^\infty \left[\exp\left\{ \frac{\pi}{4} i + \frac{\alpha_1 z (2i\Omega + \alpha)}{(4i\Omega\alpha + \alpha^2)^{1/2}} \right\} - \right.$$

$$\left. - \exp\left\{ -\frac{\pi}{4} i + \frac{\alpha_1 z (2i\Omega + \alpha)}{(4i\Omega\alpha + \alpha^2)^{1/2}} \right\} \right] \times$$

$$\times \frac{(4i\Omega\alpha + \alpha^2)^{1/4}}{i\,(2i\Omega + \alpha)^{3/2}}\, e^{-\alpha t}\, d\alpha .$$

(3.3.12).

The asymptotic expressions for v and w can be written down in the same way. The above expressions show that the flow is characterised by wave-motion. Also the contributions from the branch points die out as $t \longrightarrow \infty$ and the ultimate flow is steady and two-dimensional. Further, if $t < d_1 z/2\Omega l$ the path of integration is to be taken round the infinite semi-circle $\mathcal{R}\{s\} > 0$; since this contour does not include any singularities of the integrands, the integrals are zero. Thus it appears that the disturbances are not likely to appear far upstream.

In general, it is likely that the singularities of A_i, 'qua' functions of s, obtained from (3.2.11) and (3.2.12), all lie on the imaginary axis in the s-plane or some of them may lie on the negative real axis. The contribution from the imaginary branch points corresponds to oscillatory motion and this contribution tends to zero exponentially with time. Also the contribution from branch points with negative real part tends to zero ultimately. However, the contribution from the poles will not die out as $t \longrightarrow \infty$, and this corresponds to the steady-state solution.

The flow for $z < 0$ may be discussed on similar lines and it will be found in this case that for all z the flow is characterised by wave-motion. As the body moves, the waves generated are thus propagated downstream.

CHAPTER IV

OSCILLATION OF AXISYMMETRIC BODIES IN A ROTATING FLUID

4.1. Introduction

We consider an axisymmetric body oscillating along its axis in an inviscid, incompressible fluid rotating about the axis with uniform angular velocity Ω. The linearized equations of motion referred to a system of axes rotating with uniform angular velocity Ω, are (cf. 2.3.1)

$$\frac{\partial u'}{\partial t} - 2\Omega v' = -\frac{\partial P'}{\partial r},$$

$$\frac{\partial v'}{\partial t} + 2\Omega u' = 0,$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial P'}{\partial z},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u') + \frac{\partial w'}{\partial z} = 0, \qquad (4.1.1)$$

where

$$P' = \frac{p}{\rho} - \frac{1}{2}\Omega^2 r^2 .$$

Here we have taken the origin O at the centre of the body, OZ along the axis of rotation and r, θ polar coordinates in a plane normal to Oz. The components of fluid velocity along the directions of r, θ, z are u', $\Omega r + v'$ and w' respectively, where u', v', w' are small. Now if the motion is an oscillation of frequency $\sigma / 2\pi$, we may take

$$u' = u \exp(i\sigma t),$$

$$v' = v \exp(i\sigma t),$$

$$w' = w \exp(i\sigma t)$$

and

$$p' = p \exp(i\sigma t).$$

Now from the equations (4.1.1) we get

$$\frac{\sigma^2 - 4\Omega^2}{\sigma^2} \frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = 0. \tag{4.1.2}$$

According to the theory of second order linear differential equations the above equation is of elliptic type for $\sigma > 2\Omega$ and of hyperbolic type for $\sigma < 2\Omega$. For high frequencies ($\sigma >> 2\Omega$) the motion may be expected to resemble in general character the motion in irrotational flow, whereas for low frequencies or high angular velocities ($\sigma < 2\Omega$) there are real characteristic surfaces of revolution across which discontinuities may arise. These surfaces cut the (z, r)-

plane in lines which make an angle $\pm\ \gamma$ with the z-axis, where

$$\tan\gamma = \frac{\sigma}{(4\Omega^2 - \sigma^2)^{1/2}}$$

If the frequency of oscillation is very small ($\sigma << 2\Omega$), then the characteristic surfaces tend to become cylindrical with generators parallel to the z-axis.

Görtler[5] has pointed out the connection between oscillations of very low frequency and steady slow motion. As a slowly oscillating body passes through its position of equilibrium, it may be supposed that the flow will resemble the slow steady motion of the same body with the same velocity. In the oscillatory motion the characteristic surface corresponding to the boundary of the body will be the circumscribing cylinder $\mathcal{C}$; in the steady motion this cylinder has been shown to separate regions in which the flows are different.

Morgan[13] has investigated a number of problems of forced oscillations. Putting $(\sigma^2 - 4\Omega^2)/\sigma^2 = c^2$ in the equation (4.1.2), we find a physically admissible solution

$$P' = A\exp\left(\frac{-k}{c}z\right)J_0(kr)\exp(i\sigma t), \qquad (4.1.3)$$

where k and A are arbitrary constants, k real and positive. If $\sigma < 2\Omega$, putting $(4\Omega^2 - \sigma^2)/\sigma^2 = h^2$, we get

$$P' = A \exp\left(\frac{iK}{h} z\right) J_0(K\varkappa) \exp(i\sigma t) \tag{4.1.4}$$

If we compare the forms of solution (4.1.3) and (4.1.4) we see immediately that in the former the disturbances tend to zero with increasing z , whereas in the latter they propagate throughout the entire fluid.

Morgan was primarily interested to know as to how a steady two-dimensional motion could be set up, when the disturbance moving in the rotating liquid was three-dimensional. So, to facilitate the study of the physical phenomena involved in the problem, he considered a few particular cases of the above solutions (4.1.3) and (4.1.4) without reference to particular boundaries. Using a similarity law, he obtained, however, the solution for the case of a circular disc oscillating with frequency $\sigma > 2\Omega$.

In this chapter we consider the flow due to the oscillation of an oblate spheroid along the axis of a rotating liquid. The cases of a sphere and a circular disc are deduced from the spheroid.

4.2. Oblate spheroid oscillating along the axis

Let the section of the oblate spheroid in the $(z, \varkappa)$-plane be

$$\frac{Z^2}{a^2} + \frac{r^2}{b^2} = 1 .$$

Now equation (4.1.2) is to be solved with the following boundary conditions:

$$u \longrightarrow 0 , \quad v \longrightarrow 0 , \quad w \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty ,$$

and

$$w + \frac{a^2}{b^2} \frac{r}{Z} u = U$$

on the body, where $U \exp(i\sigma t)$ is the velocity of the body along the Z-axis.

We introduce new coordinates ξ , η defined by

$$z = \frac{c}{\sigma} (\sigma^2 - 4\Omega^2)^{1/2} \xi \eta ,$$

$$r = c (1 + \xi^2)^{1/2} (1 - \eta^2)^{1/2} , \tag{4.2.1}$$

where

$$c^2 = b^2 - \frac{a^2 \sigma^2}{\sigma^2 - 4\Omega^2} .$$

On the body we have

$$\xi = \xi_0 = \frac{a\sigma}{c (\sigma^2 - 4\Omega^2)^{1/2}} \tag{4.2.2}$$

With the above transformation equation (4.1.2) becomes

$$\frac{\partial}{\partial \xi}\left\{(1+\xi^2)\frac{\partial P}{\partial \xi}\right\} + \frac{\partial}{\partial \eta}\left\{(1-\eta^2)\frac{\partial P}{\partial \eta}\right\} = 0$$

(4.2.3)

and the boundary condition on the body becomes

$$\left(\frac{\partial P}{\partial \xi}\right)_{\xi=\xi_0} = -icU(\sigma^2 - 4\Omega^2)^{1/2}\,\eta .$$

(4.2.4)

The solution of (4.2.3) consistent with the condition at infinity is

$$P = A\eta\left[\xi \log\frac{\xi - i}{\xi + i} + 2i\right].$$

(4.2.5)

Using the boundary condition (4.2.4) we get

$$A\left(\log\frac{\xi_0 - i}{\xi_0 + i} + \frac{2i\xi_0}{\xi_0^2 + 1}\right) = -icU(\sigma^2 - 4\Omega^2)^{1/2}.$$

(4.2.6)

Now

$$\frac{\partial P}{\partial z} = \frac{h}{c^2(\xi^2+\eta^2)}\left[\xi\frac{\partial P}{\partial \xi} - \eta\frac{\partial P}{\partial \eta}\right],$$

or

$$\frac{\partial P}{\partial r} = \frac{-2iAr\eta}{c^2(\xi^2+\eta^2)(1+\xi^2)}$$

(4.2.7)

and

$$\frac{\partial P}{\partial z} = \frac{\sigma^2}{c^2(\sigma^2-4\Omega^2)}\;\frac{z}{(\xi^2+\eta^2)}\left[\frac{1+\xi^2}{\xi}\frac{\partial P}{\partial \xi} + \frac{1-\eta^2}{\eta}\frac{\partial P}{\partial \eta}\right]$$

$$= \frac{A\sigma}{c(\sigma^2-4\Omega^2)^{1/2}}\left[\log\frac{\xi-i}{\xi+i} + \frac{2i\xi}{\xi^2+\eta^2}\right].$$

(4.2.8)

Also we have from (4.2.1)

$$2c^2\xi^2 = r^2 + \frac{\sigma^2}{\sigma^2-4\Omega^2}z^2 - c^2 +$$

$$+\left[\left\{r^2 + \frac{\sigma^2}{\sigma^2-4\Omega^2}z^2 + c^2\right\}^2 - 4c^2r^2\right]^{1/2},$$

(4.2.9)

$$2c^2\eta^2 = \left[\left\{r^2 + \frac{\sigma^2}{\sigma^2-4\Omega^2}z^2 + c^2\right\}^2 - 4c^2r^2\right]^{1/2} -$$

$$-\left\{r^2 + \frac{\sigma^2}{\sigma^2-4\Omega^2}z^2 - c^2\right\}.$$

(4.2.10)

Now the expressions for the velocity components u', v', w' may be written down in terms of r and z by using the above relations and

$$u' = \mathcal{R}e \frac{i\sigma}{\sigma^2 - 4\Omega^2} \frac{\partial P}{\partial r} \exp(i\sigma t),$$

$$v' = \mathcal{R}e \frac{-2\Omega}{\sigma^2 - 4\Omega^2} \frac{\partial P}{\partial r} \exp(i\sigma t),$$

$$w' = \mathcal{R}e \frac{i}{\sigma} \frac{\partial P}{\partial z} \exp(i\sigma t), \tag{4.2.11}$$

where $\mathcal{R}e$ denotes the real part.

From (4.2.7) and (4.2.8) we find that the velocity components u', v', w' become infinite when $\xi^2 + \eta^2 = 0$, that is, when

$$r^2 + \frac{\sigma^2}{\sigma^2 - 4\Omega^2} z^2 + c^2 = 4c^2 r^2,$$

i.e., when

$$(r \pm c)^2 = \frac{\sigma^2}{4\Omega^2 - \sigma^2} z^2.$$

Thus we find that if $\sigma < 2\Omega$, there are real characteristic cones across which discontinuities arise. The sections of these cones in the (z, r)-plane are given by

$$r \pm c = \pm \frac{\sigma z}{(4\Omega^2 - \sigma^2)^{1/2}}.$$

As $\sigma \longrightarrow o$, these cones tend to cylinder with its generators parallel to the z-axis and circumscribing the body.

4.3. <u>Sphere as a limiting case of spheroid</u>

In the case of a sphere of radius a we have

$$c^2 = \frac{-4a^2\Omega^2}{\sigma^2 - 4\Omega^2},$$

$$\xi_0 = \frac{i\sigma}{2\Omega}.$$

On the axis $r = o$ we have

$$u' = o,$$

$$v' = o,$$

$$w' = \frac{U}{\log\left(\frac{\sigma - 2\Omega}{\sigma + 2\Omega}\right) + \frac{4\sigma\Omega}{\sigma^2 - 4\Omega^2}} \times$$

$$\times\left[\log\left(\frac{\sigma z - 2a\Omega}{\sigma z + 2a\Omega}\right) - \frac{4a\sigma\Omega z}{4a^2\Omega^2 - \sigma^2 z^2}\right]\cos\sigma t \qquad (4.3.1)$$

$$\text{for} \quad \sigma > 2\Omega,$$

and

$$\omega' = \frac{U}{\pi^2 + \left\{ \log\left(\frac{2\Omega - \sigma}{2\Omega + \sigma}\right) - \frac{4\sigma\Omega}{4\Omega^2 - \sigma^2} \right\}^2} \times$$

$$\times \Bigg[\Big\{ \pi^2 + \left(\log \frac{2\Omega - \sigma}{2\Omega + \sigma} - \frac{4\sigma\Omega}{4\Omega^2 - \sigma^2} \right) \left(\log \frac{2a\Omega - \sigma z}{2a\Omega + \sigma z} - \right.$$

$$\left. - \frac{4a\sigma\Omega z}{4a^2\Omega^2 - \sigma^2 z^2} \right) \Big\} \cos \sigma t -$$

$$- \pi \Big\{ \log \frac{2a\Omega - \sigma z}{2a\Omega + \sigma z} - \frac{4a\sigma\Omega z}{4a^2\Omega^2 - \sigma^2 z^2} - \log \frac{2\Omega - \sigma}{2\Omega + \sigma} +$$

$$+ \frac{4\sigma\Omega}{4\Omega^2 - \sigma^2} \Big\} \sin \sigma t \Bigg] \qquad (4.3.2)$$

$$\text{for} \quad \sigma < 2\Omega .$$

The resultant pressure on the sphere is parallel to the axis of rotation and is equal to

$$- \iint p \frac{z}{a} \, dS$$

$$= -\frac{4}{3}\pi \rho a^3 U \frac{1}{\log\left(\frac{\sigma - 2\Omega}{\sigma + 2\Omega}\right) + \frac{4\sigma\Omega}{\sigma^2 - 4\Omega^2}} \times$$

$$\times \left[\sigma \log\left(\frac{\sigma - 2\Omega}{\sigma + 2\Omega}\right) + 4\Omega \right] \sin \sigma t$$

$$\text{for} \quad \sigma > 2\Omega , \qquad (4.3.3)$$

$$= \frac{-\frac{4}{3}\pi\rho a^3 U}{\pi^2 + \left\{\log\frac{2\Omega-\sigma}{2\Omega+\sigma} - \frac{4\sigma\Omega}{4\Omega^2-\sigma^2}\right\}^2} \times$$

$$\times \left[\frac{16\pi\Omega^3}{4\Omega^2-\sigma^2}\cos\sigma t + \left\{ \pi^2\sigma + \right.\right.$$

$$\left.\left. + \left(\log\frac{2\Omega-\sigma}{2\Omega+\sigma} - \frac{4a\Omega}{4\Omega^2-\sigma^2}\right)\left(\sigma\log\frac{2\Omega-\sigma}{2\Omega+\sigma} + 4\Omega\right)\right\} \times \sin\sigma t \right] \tag{4.3.4}$$

for $\sigma < 2\Omega$.

4.4. Circular disc

The case of circular disc of radius b , with its centre on the axis of rotation and its plane perpendicular to this axis, which oscillates in a direction normal to this axis, may be considered by taking $\xi_0 = 0$ and $c = b$. Thus we have

$$P = \frac{b}{\pi} U\left(\sigma^2 - 4\Omega^2\right)^{1/2} \eta \left[\xi\log\frac{\xi - i}{\xi + i} + 2i\right] \tag{4.4.1}$$

and the velocity components u', v', w' may be written down as before. In particular on the plane $z = 0$, we have from (4.2.9) and (4.2.10)

$$\left.\begin{aligned} \xi &= \frac{(r^2 - l^2)^{1/2}}{l} \\ \eta &= 0 \end{aligned}\right\} \quad r > l \tag{4.4.2}$$

and

$$\left.\begin{aligned} \xi &= 0 \\ \eta &= \frac{(l^2 - r^2)^{1/2}}{l} \end{aligned}\right\} \quad r < l . \tag{4.4.3}$$

Therefore on $z = 0$

$$u' = 0 ,$$

$$v' = 0 ,$$

$$w' = \frac{U}{\pi}\left[\sin^{-1}\frac{l}{r} - \frac{l}{(r^2 - l^2)^{1/2}}\right]\cos\sigma t \tag{4.4.4}$$

for $r > l$, and

$$u' = \frac{2\sigma}{(\sigma^2 - 4\Omega^2)^{1/2}}\,\frac{U}{\pi}\,\frac{r}{(l^2 - r^2)^{1/2}}\cos\sigma t ,$$

$$v' = \frac{2\Omega}{(\sigma^2 - 4\Omega^2)^{1/2}}\,\frac{U}{\pi}\,\frac{r}{(l^2 - r^2)^{1/2}}\sin\sigma t ,$$

$$w' = -U\cos\sigma t \tag{4.4.5}$$

for $r < l$.

The above results agree with those obtained by Morgan[13] by another method.

SHEET NO.

CHAPTER V

SLOW MOTION OF A SPHERE IN A VISCOUS ROTATING FLUID+

5.1. Introduction

When a sphere moves along the axis of a uniformly rotating fluid, the flow produced exhibits peculiar characteristics. Although the exact equations of motion are found to be intractable for mathematical treatment, it has been found possible to ascertain some of the general features of the flow on the basis of the linearized equations. The linearized theory has shown, for example, that the flow does not become steady on the body; further, there is a singular surface $\mathcal{C}$, the cylinder with its generators parallel to the axis of rotation and circumscribing the body, which separates the regions of flow with markedly different characteristics. The assumption that the disturbance - velocities are small compared to the swirl velocity at the sphere radius gives an infinite relative angular velocity on the outer boundary of $\mathcal{C}$. Stewartson(23) has pointed out that this singularity on $\mathcal{C}$ may be removed by introducing the viscous and non-linear terms

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+ Presented at the 'Third Congress on Theoretical and Applied Mechanics', Bangalore, 1957.

near $\mathcal{C}$. But the fact that the flow does not become steady on the body (the flow continues to be oscillatory) cannot be explained by a similar reasoning.

A perfect fluid is assumed to slip over the surface of a solid body when there is a relative motion between the two. Real fluids are distinguished from a perfect fluid by the complete absence of this slip. The solution of the exact steady-state equations for a sphere moving along the axis of a rotating liquid has been made determinate by G.I.Taylor[26] by applying an additional condition of zero slip on the body. Taylor pointed out that it is possible that the solution thus obtained may represent the motion of a sphere in a rotating liquid more closely than the ordinary irrotational solution, in which allowance for slip is made, represents the actual flow in that case. Though the condition of zero slip on the body in the case of perfect fluid is artificial, it is a point of some importance because it indicates that viscosity may have to be taken into account in order to ascertain the actual flow near the sphere. The above considerations lead to the question: what is the effect of viscosity, in general, on the predictions of the linearized theory?

We consider the perturbation in the velocity of a viscous incompressible fluid rotating about an axis with uniform angular velocity Ω , due to the slow uniform motion after an impulsive start, of a sphere along the axis. When a symmetrical body moves along the axis of a rotating fluid, the flow produced is due to the interaction between the rotational motion and the motion in meridian planes. These two motions are not independent; changes in the angular velocity affect the centrifugal force and lead to motions in meridian planes and these in turn affect the rotation through the Coriolis force. In the case of inviscid fluid these two motions are related in a simple manner (cf. 3.1.2). But when viscosity is taken into account the relation is so complicated that even the linearized equations appear to be intractable for exact solution. The dynamical similarity of the flow depends on the two non-dimensional parameters R and ka, where R is the Reynold's number Ua/ν, U being the uniform velocity of the sphere, a its radius and ν the kinematic coefficient of viscosity of the fluid, and ka is the Rossby number $2\Omega a/U$. A solution valid for small values of Rka is obtained. Since the effect of viscosity is, in general, to remove any singularity in the velocity components of the fluid, no singular surface such as $\mathcal{C}$, which appears in the case of inviscid fluid, appears here. It is shown that ultimately the flow tends to become steady. The fluid resistance on the sphere is calculated and it is found that the effect of rotation of the fluid is to further increase the resistance on the sphere.

5.2. Equations of motion

We choose fixed cylindrical polar coordinates, Oz'

along the axis of rotation and r', ϕ polar coordinates in a plane normal to Oz'. The basic flow consists of a uniform angular velocity Ω about the z'-axis. A sphere of radius a impulsively starts to move along the axis at $t'=0$ with uniform velocity U. If we choose the origin of coordinates to be at the centre of the sphere, we have in effect superposed a uniform velocity $-U$ on the system and brought the sphere to rest. Let the components of the fluid velocity along r', ϕ and z' be u', $\Omega r' + v'$, w' respectively, where u', v', w' are small. Then the linearized equations of motion are

$$\frac{\partial u'}{\partial t'} - 2\Omega v' = -\frac{\partial p'}{\partial r'} + \nu\left(\nabla^2 u' - \frac{u'}{r'^2}\right), \tag{5.2.1}$$

$$\frac{\partial v'}{\partial t'} + 2\Omega u' = \nu\left(\nabla^2 v' - \frac{v'}{r'^2}\right), \tag{5.2.2}$$

$$\frac{\partial w'}{\partial t'} = -\frac{\partial p'}{\partial z'} + \nu \nabla^2 w', \tag{5.2.3}$$

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'u') + \frac{\partial w'}{\partial z'} = 0, \tag{5.2.4}$$

where

$$P' = \frac{p}{\rho} - \frac{1}{2}\Omega^2 r'^2 ,$$

p being pressure and ρ density of the fluid, and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{\partial^2}{\partial z'^2} .$$

The boundary conditions are

$$u' \to 0 , \quad v' \to 0 , \quad w' \to -U \quad \text{as } z' \to \infty$$

for fixed r', t', and on the sphere

$$u' = 0 , \quad v' = -\Omega r' , \quad w' = 0 .$$

In order to put the equations into the non-dimensional form, we write

$$u = \frac{u'}{U} , \quad v = \frac{v'}{U} , \quad w = \frac{w'}{U} ,$$

$$r = \frac{r'}{a} , \quad z = \frac{z'}{a} , \quad P = \frac{P'}{U^2} , \quad t = \frac{U}{a} t' .$$

Equations (5.2.1) to (5.2.4) are now transformed into

$$\frac{\partial u}{\partial t} - c v = -\frac{\partial P}{\partial r} + \frac{1}{R}\left(\nabla^2 u - \frac{u}{r^2}\right) , \tag{5.2.5}$$

$$\frac{\partial v}{\partial t} + c u = \frac{1}{R}\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{5.2.6}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w, \tag{5.2.7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u) + \frac{\partial w}{\partial z} = 0, \tag{5.2.8}$$

where

$$R = \frac{Ua}{\nu},$$

$$c = \frac{2\Omega a}{U}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

As Morgan(14) pointed out, since the impulsive starting of the sphere gives rise to finite velocities at $t = 0$, the particle displacements at $t = 0$ are still zero and so the rotational character of the motion has not yet become evident. Therefore the initial motion relative to the rotating system may be taken to be the irrotational motion with the given boundary conditions. Taking the velocity potential of this irrotational flow to be $\chi(r, z)$, we have at $t = 0$

$$u = \frac{\partial \chi}{\partial r}, \quad v = 0, \quad w = \frac{\partial \chi}{\partial z}. \tag{5.2.9}$$

Now to take the Laplace transforms of u, v, w and P, we put

$$\bar{u} = \int_0^\infty \exp(-st)\, u(r, z, t)\, dt \qquad etc.$$

Now equations (5.2.5) to (5.2.8) become

$$s\bar{u} - c\bar{v} = -\frac{\partial \bar{N}}{\partial r} + \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)\bar{u}, \tag{5.2.10}$$

$$s\bar{v} + c\bar{u} = \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)\bar{v}, \tag{5.2.11}$$

$$s\bar{w} = -\frac{\partial \bar{N}}{\partial z} + \frac{1}{R}\nabla^2 \bar{w}, \tag{5.2.12}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0, \tag{5.2.13}$$

where

$$\bar{N} = \bar{P} - \chi ,$$

and the boundary conditions become

$$\bar{u} \longrightarrow 0 , \quad \bar{v} \longrightarrow 0 , \quad \bar{w} \longrightarrow -\frac{1}{s} \quad as \quad z \longrightarrow \infty ,$$

and

$$\bar{u} = 0 , \quad \bar{v} = -\frac{a\Omega r}{U s} , \quad \bar{w} = 0$$

on the sphere.

Eliminating $\bar{N}$ between (5.2.10) and (5.2.12), we get

$$\left(\nabla^2 - \frac{1}{r^2} - Rs\right)\left(\frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial r}\right) + cR\,\frac{\partial \bar{v}}{\partial z} = 0 . \tag{5.2.14}$$

Putting

$$\bar{u} = \frac{1}{r}\,\frac{\partial \psi}{\partial z} ,$$

$$\bar{w} = -\frac{1}{r}\,\frac{\partial \psi}{\partial r} , \tag{5.2.15}$$

equation (5.2.13) is satisfied and (5.2.14) becomes

$$\left(D^2 - Rs\right) D^2 \psi + c_1\,\frac{\partial V}{\partial z} = 0 , \tag{5.2.16}$$

where

$$C_1 = c R = \frac{2 \Omega a^2}{\nu} ,$$

$$V = r \bar{v}$$

and

$$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} .$$

Also (5.2.11) can be written as

$$\left(D^2 - R \delta \right) V - C_1 \frac{\partial \psi}{\partial z} = 0 .$$

(5.2.17)

Equations (5.2.16) and (5.2.17) express the relation between the rotational motion represented by V and the motion in meridian planes represented by ψ. It appears that these two equations cannot be solved exactly even in the case of a sphere. However, a solution for small values of C_1 may be obtained by the method of successive approximation. Thus assuming C_1 to be small, we may take

$$\psi = \psi_0 + C_1 \psi_1 + C_1^2 \psi_2 + \cdots \cdots ,$$

$$V = V_0 + C_1 V_1 + C_1^2 V_2 + \cdots \cdots ,$$

(5.2.18)

where $\psi_0 , \psi_1 , \psi_2 , \cdots \cdots$ and $V_0 , V_1 , V_2 , \cdots$ are functions of r and z.

Substituting for ψ and V from (5.2.18) in (5.2.16) and (5.2.17) and equating the coefficients of equal powers of C_1, we get

$$(D^2 - R\delta)\, D^2 \psi_0 = 0 , \tag{5.2.19}$$

$$(D^2 - R\delta)\, D^2 \psi_1 + \frac{\partial V_0}{\partial z} = 0 , \quad \text{etc.}, \tag{5.2.20}$$

$$(D^2 - R\delta)\, V_0 = 0 , \tag{5.2.21}$$

$$(D^2 - R\delta)\, V_1 - \frac{\partial \psi_0}{\partial z} = 0 , \quad \text{etc.} \tag{5.2.22}$$

If we restrict ourselves to the first order solution for V, that is, a solution containing first power of C_1, and a second order solution for ψ, the above equations are to be solved for ψ_0, ψ_1, V_0 and V_1.

5.3. Solution to the problem

Hereafter we take (r, θ, ϕ) to denote spherical polar coordinates. In these coordinates equation (5.2.21) becomes

$$\frac{\partial^2 V_0}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial V_0}{\partial\theta}\right) + h^2 V_0 = 0 ,$$

(5.3.1)

where $h^2 = -R s$. The appropriate solution of (5.3.1) is

$$V_0 = A_0 \, x^{1/2} \, H^{(2)}_{3/2}(x) \, \sin^2\theta ,$$

(5.3.2)

where $x = hr$, $H^{(2)}_{3/2}(x)$ is the second Hankel function of order $3/2$ and A_0 is an arbitrary constant. Using the boundary condition on the sphere, viz.

$$V_0 = -\frac{a\Omega}{Us}\sin^2\theta$$

when $r = 1$, we get

$$V_0 = -\frac{a\Omega}{Us}\frac{1}{r}\left\{\frac{1 + r(Rs)^{1/2}}{1 + (Rs)^{1/2}}\right\} \times$$

$$\times \exp\left\{-(r-1)(Rs)^{1/2}\right\} \sin^2\theta .$$

(5.3.3)

Again, the appropriate solution of (5.2.19) is

$$\psi_0 = \left[B_0 r^2 + \frac{C_0}{r} + D_0 \left(1 + \frac{1}{ihr}\right) \exp(-ihr) \right] \sin^2\theta .$$

(5.3.4)

The boundary condition at infinity gives $B_0 = 1/2S$, and the boundary condition on the sphere, viz.

$$\bar{u} = 0 \quad , \quad \bar{w} = 0$$

when $r = 1$, gives

$$C_0 = -\frac{1}{2S}\left(1 + \frac{3}{ih} - \frac{3}{h^2}\right)$$

and

$$D_0 = \frac{3}{2S} \frac{\exp(ih)}{ih} .$$

So we get

$$\psi_0 = \left[\frac{r^2}{2S} - \frac{1}{2S}\left\{1 + \frac{3}{(RS)^{1/2}} + \frac{3}{RS}\right\}\frac{1}{r} + \right.$$

$$\left. + \frac{3}{2S} \frac{\exp\{-(r-1)(RS)^{1/2}\}}{(RS)^{1/2}} \left\{1 + \frac{1}{r(RS)^{1/2}}\right\} \right] \sin^2\theta . \tag{5.3.5}$$

Substituting the value of V_0 from (5.3.2), equation (5.2.20) becomes

$$D^4\psi_1 + h^2 D^2 \psi_1$$

$$= A_0 h\, x^{1/2} H^{(2)}_{5/2}(x) \sin^2\theta \cos\theta \,,.$$

(5.3.6).

where

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

Putting $r = x/h$ and $\psi_1 = x^{1/2} F(x) \sin^2\theta \cos\theta$,

(5.3.6) takes the form

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{25}{4}\frac{1}{x^2}\right)\left\{\frac{d^2F}{dx^2} + \frac{1}{x}\frac{dF}{dx} + \left(1 - \frac{25}{4}\frac{1}{x^2}\right)F\right\}$$

$$= \frac{A_0}{h^3} H^{(2)}_{5/2}(x).$$

(5.3.7)

The solution of the above equation is

$$F = A\, x^{5/2} + \frac{B}{x^{5/2}} + \mathcal{C}\, H^{(2)}_{5/2}(x) + \mathcal{D}\, H^{(1)}_{5/2}(x),$$

(5.3.8)

where $H^{(1)}_{5/2}(x)$ is the first Hankel function, and $A, B, \mathcal{C}, \mathcal{D}$ are functions of x to be determined. Using the method of variation of parameters, we get[30].

$$A = \frac{A_0}{10h^3}\int x^{-3/2}\, H^{(2)}_{5/2}(x)\, dx$$

$$= -\frac{A_0}{10h^3}\, x^{-3/2}\, H^{(2)}_{3/2}(x)$$

$$= \frac{A_0}{10h^3}\left(\frac{\pi}{2}\right)^{-1/2}\frac{1}{x^2}\left(1+\frac{1}{ix}\right)\exp(-ix),$$

$$B = -\frac{A_0}{10h^3}\int x^{7/2}\, H^{(2)}_{5/2}(x)\, dx$$

$$= -\frac{A_0}{10h^3}\, x^{7/2}\, H^{(2)}_{7/2}(x)$$

$$= -\frac{A_0}{10h^3}\left(\frac{\pi}{2}\right)^{-1/2} x^3\left(1+\frac{6}{ix}-\frac{15}{x^2}-\frac{15}{ix^3}\right)\exp(-ix),$$

$$\mathcal{C} = \frac{\pi i}{8}\,\frac{A_0}{h^3}\int x\, H^{(2)}_{5/2}(x)\, H^{(1)}_{5/2}(x)\, dx$$

$$= \frac{\pi i}{32}\,\frac{A_0}{h^3}\, x^2\Big[2\, H^{(1)}_{5/2}(x)\, H^{(2)}_{5/2}(x) - H^{(1)}_{3/2}(x)\, H^{(2)}_{7/2}(x) - H^{(1)}_{7/2}(x)\, H^{(2)}_{3/2}(x)\Big]$$

$$= \frac{iA_0}{4h^3}\left(x-\frac{3}{x}-\frac{3}{x^3}\right),$$

$$\mathcal{D} = -\frac{\pi i}{8}\,\frac{A_0}{h^3}\int x\left\{H^{(2)}_{5/2}(x)\right\}^2 dx$$

$$= -\frac{\pi i}{16}\,\frac{A_0}{h^3}\,x^2\left[\left\{H^{(2)}_{5/2}(x)\right\}^2 - H^{(2)}_{3/2}(x)\,H^{(2)}_{7/2}(x)\right].$$

$$= \frac{i A_0}{8h^3}\left(i + \frac{6}{x} + \frac{12}{ix^2} - \frac{6}{x^3}\right)\exp(-2ix).$$

Thus we get

$$\psi_1 = \frac{A_0}{ih^3}\left(\tfrac{1}{2}\pi\right)^{-1/2}\left[\frac{B_1}{x^2} + \left\{\left(1+\frac{3}{ix}-\frac{3}{x^2}\right)\left(\frac{ix}{4}-\frac{3}{8}+\mathcal{C}_1\right) + \right.\right.$$

$$\left.\left. + \frac{3}{2ix}\left(\frac{1}{2}+\frac{1}{ix}\right)\right\}\exp(-ix)\right]\sin^2\theta\cos\theta\,, \tag{5.3.9}$$

where B_1 and $\mathcal{C}_1$ are arbitrary constants.

Using the boundary condition on the body, viz.

$$\bar{u} = 0\,, \quad \bar{w} = 0$$

when $x = h$, we find

$$B_1 = \frac{\exp(-ih)}{4(1+ih)}\,h^2\left(ih + 6 + \frac{12}{ih} - \frac{6}{h^2}\right)$$

and

$$\mathcal{C}_1 = -\left\{\frac{3}{8} - \frac{ih}{4} + \frac{ih}{4(1+ih)}\right\}.$$

So we get

$$\psi_1 = -\frac{\Omega\,\nu}{U^2}\Bigg[\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{s^2\{1+(Rs)^{1/2}\}}\Big\{\Big(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2 Rs}\Big)\times$$

$$\times\Big(\frac{1}{2}-\frac{Rs}{4(1+(Rs)^{1/2})}\Big)+\frac{1}{4}\big(1+r(Rs)^{1/2}\big)\Big\} -$$

$$-\frac{\Big\{(Rs)^{1/2}+6+\dfrac{12}{(Rs)^{1/2}}+\dfrac{6}{Rs}\Big\}}{4r^2s^2\{1+(Rs)^{1/2}\}^2}\Bigg]\sin^2\theta\cos\theta. \tag{5.3.10}$$

On substituting the value of ψ_0 from (5.3.4), equation (5.2.22) becomes

$$\frac{\partial^2 V_1}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial V_1}{\partial\theta}\Big)+h^2V_1$$

$$= -h^2\Big\{\frac{3\,\mathcal{C}_0}{x^2}-\Big(\frac{\pi}{2}\Big)^{1/2}\frac{\mathcal{D}_0}{h}\,x^{1/2}H^{(2)}_{5/2}(x)\Big\}\sin^2\theta\cos\theta. \tag{5.3.11}$$

Putting

$$V_1 = x^{1/2}\,G(x)\,\sin^2\theta\cos\theta,$$

(5.3.11) takes the form

$$\frac{d^2G}{dx^2}+\frac{1}{x}\frac{dG}{dx}+\Big(1-\frac{25}{4}\frac{1}{x^2}\Big)G$$

$$= -\Big\{\frac{3\,\mathcal{C}_0}{x^{5/2}}-\Big(\frac{\pi}{2}\Big)^{1/2}\frac{\mathcal{D}_0}{h}H^{(2)}_{5/2}(x)\Big\}. \tag{5.3.12}$$

By using the same method as before, the appropriate solution of (5.3.11) is found to be

$$V_1 = \frac{3}{2R}\Big[-\frac{1}{s^2}\Big\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\Big\}\frac{1}{r^2}+\Big\{3+\frac{6}{r(Rs)^{1/2}}+$$

$$+\frac{6}{r^2Rs}+r(Rs)^{1/2}-\frac{\left(1+(Rs)^{1/2}\right)\left(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2Rs}\right)}{\left(1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right)}\Big\}\times$$

$$\times\frac{1}{2s^2}\exp\{-(r-1)(Rs)^{1/2}\}\Big]\sin^2\theta\cos\theta. \tag{5.3.13}$$

Thus we have

$$\psi = \Big[\frac{r^2}{2s}-\frac{1}{2s}\Big\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\Big\}\frac{1}{r}+$$

$$+\frac{3}{2s}\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{(Rs)^{1/2}}\Big\{1+\frac{1}{r(Rs)^{1/2}}\Big\}\Big]\sin^2\theta -$$

$$-\frac{2a^2\Omega^2}{U^2}\Big[\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{s^2(1+(Rs)^{1/2})}\Big\{\Big(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2Rs}\Big)\times$$

$$\times\Big(\frac{1}{2}-\frac{Rs}{4(1+(Rs)^{1/2})}\Big)+\frac{1}{4}\left(1+r(Rs)^{1/2}\right)\Big\}-$$

$$-\frac{\{(Rs)^{1/2}+6+\frac{12}{(Rs)^{1/2}}+\frac{6}{Rs}\}}{4r^2s^2\{1+(Rs)^{1/2}\}^2}\Big]\sin^2\theta\cos\theta \tag{5.3.14}$$

and

$$V = -\frac{a\Omega}{U s}\,\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{r}\left\{\frac{1+r(Rs)^{1/2}}{1+(Rs)^{1/2}}\right\}\sin^2\theta \;+$$

$$+\frac{3a\Omega}{U}\Bigg[-\frac{1}{s^2}\left\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right\}\frac{1}{r^2}+\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{2s^2}\times$$

$$\times\Bigg\{3+\frac{6}{r(Rs)^{1/2}}+\frac{6}{r^2Rs}+r(Rs)^{1/2}-$$

$$-\frac{\left(1+(Rs)^{1/2}\right)\left(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2Rs}\right)}{\left(1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right)}\Bigg\}\Bigg]\sin^2\theta\cos\theta. \tag{5.3.15}$$

If we now denote by u_1 the velocity along the radius vector, by w_1 the velocity at right angles to it, in a meridian plane, and by v_1 the azimuthal velocity, in non-dimensional form, we have

$$\bar{u}_1 = \Bigg[\frac{1}{s}-\frac{1}{s}\left\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right\}\frac{1}{r^3}\;+$$

$$+\frac{3}{r^2 s}\,\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{(Rs)^{1/2}}\left\{1+\frac{1}{r(Rs)^{1/2}}\right\}\Bigg]\cos\theta\;-$$

$$
- \frac{2a^2\Omega^2}{U^2 r^2}\left[\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{s^2\{1+(Rs)^{1/2}\}}\left\{\frac{1}{4}\left(1+r(Rs)^{1/2}\right) + \left(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2Rs}\right)\left(\frac{1}{2}-\frac{Rs}{4(1+(Rs)^{1/2})}\right)\right\} - \frac{\left\{(Rs)^{1/2}+6+\frac{12}{(Rs)^{1/2}}+\frac{6}{Rs}\right\}}{4r^2s^2\left(1+(Rs)^{1/2}\right)^2}\right](2-3\sin^2\theta),
$$

(5.3.16)

$$
\bar{w}_1 = -\frac{1}{r}\left[\frac{r}{s}+\frac{1}{2s}\left\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right\}\frac{1}{r^2} - \frac{3}{2s}\,\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{(Rs)^{1/2}}\left\{(Rs)^{1/2}+\frac{1}{r}+\frac{1}{r^2(Rs)^{1/2}}\right\}\right]\sin\theta + \frac{2a^2\Omega^2}{U^2 r}\left[\frac{-\exp\{-(r-1)(Rs)^{1/2}\}}{s^2\left(1+(Rs)^{1/2}\right)}\left\{\left((Rs)^{1/2}+\frac{3}{r}+\frac{6}{r^2(Rs)^{1/2}}+\frac{6}{r^3Rs}\right)\left(\frac{1}{2}-\frac{Rs}{4(1+(Rs)^{1/2})}\right)+\frac{1}{4}rRs\right\} + \frac{\left\{(Rs)^{1/2}+6+\frac{12}{(Rs)^{1/2}}+\frac{6}{Rs}\right\}}{2r^3s^2\left\{1+(Rs)^{1/2}\right\}^2}\right]\sin\theta\cos\theta,
$$

(5.3.17)

$$\bar{v}_1 = -\frac{a\Omega}{Us}\frac{1}{r^2}\left\{\frac{1+r(Rs)^{1/2}}{1+(Rs)^{1/2}}\right\}\exp\{-(r-1)(Rs)^{1/2}\}\sin\theta +$$

$$+\frac{3a\Omega}{Ur}\left[-\frac{1}{s^2}\left\{1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right\}\frac{1}{r^2}+\frac{\exp\{-(r-1)(Rs)^{1/2}\}}{2s^2}\times\right.$$

$$\times\left\{3+\frac{6}{r(Rs)^{1/2}}+\frac{6}{r^2 Rs}+r(Rs)^{1/2}-\right.$$

$$\left.\left.-\frac{\left(1+(Rs)^{1/2}\right)\left(1+\frac{3}{r(Rs)^{1/2}}+\frac{3}{r^2 Rs}\right)}{\left(1+\frac{3}{(Rs)^{1/2}}+\frac{3}{Rs}\right)}\right\}\right]\sin\theta\cos\theta . \tag{5.3.18}$$

By taking the inverse transforms of $\bar{u}_1$, $\bar{v}_1$, $\bar{w}_1$ we can find the components of the fluid velocity at any instant. For points near the sphere, we find, as $t \to \infty$,

$$u_1 = \left(1+\frac{1}{2r^3}-\frac{3}{2r}\right)\cos\theta + \frac{a^3\Omega^2}{8U\nu}\left(1-\frac{1}{r^2}\right)\left(2-3\sin^2\theta\right), \tag{5.3.19}$$

$$w_1 = \left(-1+\frac{1}{4r^3}+\frac{3}{4r}\right)\sin\theta - \frac{a^3\Omega^2}{8U\nu}\left(1-\frac{1}{r^4}\right)\sin\theta\cos\theta , \tag{5.3.20}$$

$$v_1 = -\frac{a\Omega}{Ur^2}\sin\theta - \frac{a^2\Omega}{8\nu}\left(3r-\frac{2}{r}-\frac{1}{r^3}\right)\sin\theta\cos\theta . \tag{5.3.21}$$

Thus we see that the flow tends to steady state ultimately. Now if v_r, v_θ, v_ϕ denote the actual velocities along r, θ, ϕ respectively, we have

$$v_r = U\left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right)\cos\theta + \frac{a^3\Omega^2}{8\nu}\left(1 - \frac{a^2}{r^2}\right)\left(2 - 3\sin^2\theta\right), \tag{5.3.22}$$

$$v_\theta = U\left(-1 + \frac{a^3}{4r^3} + \frac{3a}{4r}\right)\sin\theta - \frac{a^3\Omega^2}{8\nu}\left(1 - \frac{a^4}{r^4}\right)\sin\theta\cos\theta, \tag{5.3.23}$$

$$v_\phi = \Omega\left(r - \frac{a^3}{r^2}\right)\sin\theta - \frac{a^2\Omega U}{8\nu}\left(\frac{3r}{a} - \frac{2a}{r} - \frac{a^3}{r^3}\right)\sin\theta\cos\theta. \tag{5.3.24}$$

5.4. Resistance on the sphere

The force on the sphere may be now calculated from the stress-formulae[4]

$$p_{rr} = -p + \mu e_{rr},$$

$$p_{\theta\theta} = -p + \mu e_{\theta\theta},$$

$$p_{\phi\phi} = -p + \mu e_{\phi\phi},$$

$$p_{r\theta} = p_{\theta r} = \mu\, e_{r\theta},$$

$$p_{\theta\phi} = p_{\phi\theta} = \mu\, e_{\theta\phi},$$

$$p_{\phi r} = p_{r\phi} = \mu\, e_{\phi r},$$

where

$$\frac{1}{2} e_{rr} = \frac{\partial v_r}{\partial r},$$

$$\frac{1}{2} e_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r},$$

$$\frac{1}{2} e_{\phi\phi} = \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r},$$

$$e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta},$$

$$e_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right),$$

$$e_{\phi r} = r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right).$$

We find, on the sphere

$$e_{rr} = 0, \quad e_{\theta\theta} = 0,$$

$$e_{\phi\phi} = 0, \quad e_{\theta\phi} = 0,$$

$$e_{\phi r} = 3\Omega\sin\theta - \frac{a\Omega U}{\nu}\sin\theta\cos\theta,$$

$$e_{r\theta} = -\frac{1}{2}\frac{a^2\Omega^2}{\nu}\sin\theta\cos\theta.$$

The resultant force is found to be

$$\frac{\pi^2}{2} \rho a^3 U \Omega + 6 \pi \rho \nu a U + \frac{\pi^2}{4} \rho a^4 \Omega^2 + $$
$$+ \frac{2}{3} \frac{\pi \rho a^5 \Omega^2 U}{\nu} \qquad (5.4.1)$$

along the direction of Z -negative.

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Slow motion of a paraboloid of revolution in a rotating fluid

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SUMMARY

The slow uniform motion, after an impulsive start from relative rest, of a paraboloid of revolution along the axis of a rotating fluid is investigated by using a perturbation method. The principal purpose of the note is to illustrate the mechanism by which the fluid is not subjected to any substantial radial displacement, which is a direct consequence of the requirement that the circulation round material circuits should be constant when the perturbation velocities due to the motion of the paraboloid remain small. It appears that the mechanism is an oscillatory one in which the distance between any fluid particle and the axis of rotation oscillates sinusoidally in time with small amplitude. As time progresses, the amplitude of the oscillation decays to zero everywhere except on the paraboloid. The ultimate motion is then a rigid body rotation everywhere except on the paraboloid and the axis of rotation, where the perturbation velocities continue to oscillate indefinitely with small amplitude.

1. INTRODUCTION

The motion of bodies in a rotating fluid has been a subject for a series of investigations in recent years. The perturbation caused by the motion of a body in an inviscid fluid exhibits different characteristics according as the fluid is at rest at infinity or is rotating about an axis there. Thus if the fluid is at rest at infinity, the flow is everywhere irrotational and dependent only on the instantaneous velocity of the body. But if the fluid is rotating about an axis, the perturbation in the fluid velocity depends not only on the instantaneous velocity of the body but also on its past history and is in general neither steady nor irrotational; and even in cases where a steady solution of the governing equation can be found, there is no guarantee that the flow can be set up by starting the body from rest relative to the rotating system. For these reasons it is necessary to consider an initial-value problem while dealing with this type of fluid motion.

When the body moves slowly it has been customary to use a small perturbation theory. Using this method Stewartson (1952, 1953) has investigated the slow uniform motion, after an impulsive start from relative rest, of a sphere and an ellipsoid along the axis of a rotating liquid. In both these cases he found that ultimately the fluid inside the circumscribing cylinder C with its generators parallel to the axis of rotation is pushed along in front of the body as if it were solid, while outside the cylinder there is a shearing motion parallel to the axis of rotation. There is also a swirling motion about the axis inside the cylinder C. The ultimate velocity distribution in the fluid is in general steady and two-dimensional (in the sense that the motion is the same in all planes perpendicular to the axis of rotation) everywhere except on the body and on its axis, where it oscillates finitely. In fact the linearized equations show that every slow and steady motion must also be two-dimensional.

There is, however, an *a priori* difficulty with any theory which supposes that the perturbation remains small. Since the circulation round any material circular circuit concentric with the axis must remain constant, the radius of such a circuit must always be nearly equal to its initial radius. At first sight this restriction on the total strain of the fluid seems unlikely to be satisfied since it is inconsistent with any prolonged general streaming (however small) past the body. Nevertheless, Stewartson's (1952, 1953) solutions do show that the perturbation can remain small, and these solutions must therefore contain an explanation of the mechanism by which the circulation is maintained at a constant level, though Stewartson does not point this out. Moreover, his solution is very complicated, largely due to the formation of the singular surface at the cylinder C, and this rather obscures the mechanism of the flow. In this note, therefore, a much simpler solution which does not have a surface corresponding to C is obtained, the primary aim being to illustrate the mechanism by which the circulation remains constant even when the perturbation remains small.

The flow considered is that due to the slow uniform motion, started impulsively from relative rest, of a paraboloid of revolution along the axis of a rotating liquid. It is found that the radius of any material circuit concentric with the axis of rotation executes small oscillations which are 180° out of phase with the corresponding oscillations in the azimuthal velocity component. As time progresses, the amplitude of these oscillations decays to zero everywhere except on the paraboloid. The ultimate flow is then steady and two-dimensional everywhere except on the paraboloid and on the axis of rotation. On the paraboloid the velocity oscillates finitely and on the axis the velocity component parallel to the axis oscillates finitely and the other components are zero. The swirling motion about the axis found in the case of the sphere and ellipsoid is absent here.

It may be pointed out that the results obtained here can be deduced from Stewartson's (1953) solution for an ellipsoid by carrying out the usual limiting process. But the procedure adopted here is found to be simpler.

2. Solution to the problem

We choose cylindrical polar coordinates, Oz along the axis of rotation and (r, θ) polar coordinates in a plane normal to Oz. Let the unperturbed motion of the fluid consist of a uniform angular velocity Ω about the z-axis.

A paraboloid of revolution (whose axis of symmetry coincides with the z-axis) impulsively starts to move along the axis at $t=0$ with uniform velocity V. If we choose the origin of coordinates to be in the body, we have in effect superposed a uniform velocity $-V$ on the system and brought the body to rest. Let the components of the fluid velocity along the directions of increasing r, θ, z, be u, $\Omega r+v$, w, respectively, where u, v, w are small. Then the linearized equations of motion are

$$\left.\begin{aligned} \frac{\partial u}{\partial t}-2\Omega v &= -\frac{\partial P}{\partial r}, \\ \frac{\partial v}{\partial t}+2\Omega u &= 0, \\ \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z}, \end{aligned}\right\} \tag{2.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0, \tag{2.2}$$

where
$$P=\frac{p}{\rho}-\frac{1}{2}\Omega^2r^2.$$

The boundary conditions are that

$$u\to 0, \qquad v\to 0, \qquad w\to -V, \qquad \text{as } z\to\infty \text{ for fixed } r,\, t, \tag{2.3}$$

and, on the body, the component of the fluid velocity normal to the body is zero.

As Morgan (1953) pointed out, the initial disturbance travels with infinite velocity and the initial motion relative to the rotating system must be the irrotational motion with the given boundary conditions. Taking the velocity potential of this irrotational flow to be

$$\phi(r,z) = -Vz+\chi(r,z),$$

we have at $t=0$

$$u=\frac{\partial\chi}{\partial r}, \qquad v=0, \qquad w=-V+\frac{\partial\chi}{\partial z}.$$

Now to take the Laplace transforms of u, v, w, and P, we put

$$\bar{u}=\int_0^\infty e^{-st}u(r,z,t)\,dt, \quad \text{etc.}$$

Then (2.1) and (2.2) become

$$\left.\begin{aligned} s\bar{u}-2\Omega\bar{v} &= -\frac{\partial\bar{N}}{\partial r}, \\ s\bar{v}+2\Omega\bar{u} &= 0, \\ s\bar{w}+V &= -\frac{\partial\bar{N}}{\partial z}, \end{aligned}\right\} \tag{2.4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u})+\frac{\partial\bar{w}}{\partial z}=0, \tag{2.5}$$

where
$$\overline{N} = \overline{P} - \chi,$$
and the boundary conditions (2.3) become
$$\bar{u} \to 0, \qquad \bar{v} \to 0, \qquad \bar{w} \to -\frac{V}{s} \qquad \text{as } z \to \infty. \tag{2.6}$$
Solving (2.4) we get
$$\left.\begin{aligned} \bar{u} &= -\frac{s}{s^2+4\Omega^2}\frac{\partial \overline{N}}{\partial r}, \\ \bar{v} &= \frac{2\Omega}{s^2+4\Omega^2}\frac{\partial \overline{N}}{\partial r}, \\ \bar{w} &= -\frac{V}{s} - \frac{1}{s}\frac{\partial \overline{N}}{\partial z}, \end{aligned}\right\} \tag{2.7}$$
so that the continuity condition (2.5) becomes
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{N}}{\partial r}\right) + \frac{s^2+4\Omega^2}{s^2}\frac{\partial^2 \overline{N}}{\partial z^2} = 0, \tag{2.8}$$
and the boundary condition (2.6) becomes
$$\frac{\partial \overline{N}}{\partial z} \to 0 \quad \text{as } z \to \infty. \tag{2.9}$$

If the section of the paraboloid in the (z, r)-plane is $z = -ar^2$, the condition on the body is
$$2aru + w = 0, \qquad \text{or } 2ar\bar{u} + \bar{w} = 0,$$
or, in view of (2.7),
$$2ar\frac{s^2}{s^2+4\Omega^2}\frac{\partial \overline{N}}{\partial r} + \frac{\partial \overline{N}}{\partial z} = -V. \tag{2.10}$$
So equation (2.8) is to be solved with boundary conditions (2.9) and (2.10).

Now we can easily formulate the problem in a coordinate system in which (2.8) can be solved simply and in which the body is a coordinate surface, by taking a suitable transformation of independent variables. We introduce new coordinates (ξ, η) defined by
$$z + \frac{K^2}{4a} = K(\xi^2 - \eta^2), \qquad r = 2\xi\eta, \tag{2.11}$$
where
$$K^2 = \frac{s^2+4\Omega^2}{s^2}.$$
On the paraboloid we have
$$\xi = \xi_0 = (K/4a)^{1/2}.$$
With the above transformation equation (2.8) becomes
$$\frac{\partial^2 \overline{N}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \overline{N}}{\partial \xi} + \frac{\partial^2 \overline{N}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \overline{N}}{\partial \eta} = 0, \tag{2.12}$$
and the boundary condition (2.10) becomes
$$\frac{\partial \overline{N}}{\partial \xi} = -2KV\xi_0 \quad \text{on } \xi = \xi_0. \tag{2.13}$$

The appropriate solution of (2.12) is

$$\bar{N} = (A + B\log\xi)(C + D\log\eta),$$

and using (2.13) we get

$$\bar{N} = -2KV\xi_0 \log\xi. \tag{2.14}$$

Now

$$\frac{\partial \bar{N}}{\partial z} = \frac{1}{2K(\xi^2+\eta^2)}\left(\xi\frac{\partial \bar{N}}{\partial \xi} - \eta\frac{\partial \bar{N}}{\partial \eta}\right) = -\frac{V\xi_0^2}{\xi^2+\eta^2}.$$

Therefore $\partial\bar{N}/\partial z \to 0$ as $\xi \to \infty$, in agreement with (2.9). Thus (2.14) is the appropriate solution.

The results for $\bar{u}$, $\bar{v}$, $\bar{w}$ follow immediately from (2.7). Finally, inverting these Laplace transforms, we find that the velocity components at any point of the fluid are given by

$$\left.\begin{aligned} u &= \frac{V}{8ar\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(1 - \frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right)\frac{e^{st}}{s}\,ds, \\ v &= -\frac{\Omega V}{4ar\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(1 - \frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right)\frac{e^{st}}{s^2}\,ds, \\ w &= -V + \frac{V}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^2+4\Omega^2}{\omega^2}\,e^{st}\,ds, \end{aligned}\right\} \tag{2.15}$$

where $\gamma > 0$ and

$$\left.\begin{aligned} \omega^2 &= \{(1+4az)^2+16a^2r^2\}^{1/2}(s^2+4\Omega^2 l_1^2)^{1/2}(s^2+4\Omega^2 l_2^2)^{1/2}, \\ l_1^2 &= \frac{1+4az+8a^2r^2-8ar(a^2r^2+az)^{1/2}}{(1+4az)^2+16a^2r^2}, \\ l_2^2 &= \frac{1+4az+8a^2r^2+8ar(a^2r^2+az)^{1/2}}{(1+4az)^2+16a^2r^2}. \end{aligned}\right\} \tag{2.16}$$

These results represent a complete formal solution to the problem. For the present purpose of ascertaining the general features of the flow, however, it is only necessary to consider certain special cases of the formulae (2.15).

3. General features of the flow

On the surface of the paraboloid the integrals (2.15) simplify considerably and it is possible to evaluate them in the forms

$$\left.\begin{aligned} u &= \frac{2Var}{1+4a^2r^2}\cos\frac{2\Omega t}{(1+4a^2r^2)^{1/2}}, \\ v &= -\frac{2Var}{(1+4a^2r^2)^{1/2}}\sin\frac{2\Omega t}{(1+4a^2r^2)^{1/2}}, \\ w &= -\frac{4Va^2r^2}{1+4a^2r^2}\cos\frac{2\Omega t}{(1+4a^2r^2)^{1/2}}. \end{aligned}\right\} \tag{3.1}$$

Thus we find that the motion never becomes steady on the paraboloid. More important, perhaps, these results show very simply the way in which the circulation round circular material circuits, concentric with the axis of rotation and lying on the paraboloid, remains constant. Since the radial velocity u oscillates sinusoidally in time, the radius of such a circuit must oscillate in a similar way, and the primary rotation then makes an oscillatory contribution to the circulation round the circuit. This must, in turn, be counterbalanced by an oscillatory contribution from the azimuthal perturbation velocity v, in accordance with (3.1). As far as the validity of the linearized analysis is concerned, the essential feature of this mechanism is that no fluid particle is displaced appreciably in a radial direction from its initial position.

Similarly, on the axis of rotation we find

$$u = 0, \qquad v = 0, \qquad w = -\frac{4Vaz}{1+4az}\cos\frac{2\Omega t}{(1+4az)^{1/2}}. \tag{3.2}$$

Here again the oscillatory axial velocity implies that small material circuits surrounding the axis of rotation are never swept on to the surface of the paraboloid, thereby increasing their perimeter by a large factor; an essential result if the azimuthal perturbation velocity is to remain small.

From these special cases it is reasonable to infer that the same oscillatory mechanism is responsible for maintaining the radial positions of all fluid particles, and this may be verified directly when the motion is approaching its ultimate form. Thus, for large values of t, the integrals in (2.15) may be evaluated by inserting cuts in the s-plane from $s = \pm 2i\Omega l_1$ and $s = \pm 2i\Omega l_2$ along lines on which the imaginary part of s is constant and the real part decreases. The path of integration may now be replaced by a path round the infinite semicircle $\mathscr{R}\{s\} < 0$ and round the four cuts. For example, the contribution from the branch point $s = 2i\Omega l_1$ to the integral in the first of the equations (2.15) is found to be

$$e^{2i\Omega l_1 t}\frac{1-(1+4az)l_1^2}{l_1(l_2^2-l_1^2)^{1/2}(4i\Omega l_1)^{1/2}}\frac{\Gamma(\frac{1}{2})}{t^{1/2}}$$

for large t. In this way we find that

$$u \sim -\frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)l_1^2}{l_1(\Omega l_1)^{1/2}}\sin(2\Omega l_1 t - \tfrac{1}{4}\pi) + \right.$$
$$\left. + \frac{1-(1+4az)l_2^2}{l_2(\Omega l_2)^{1/2}}\sin(2\Omega l_2 t + \tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}},$$

$$v \sim \frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)l_1^2}{l_1^2(\Omega l_1)^{1/2}}\cos(2\Omega l_1 t - \tfrac{1}{4}\pi) + \right.$$
$$\left. + \frac{1-(1+4az)l_2^2}{l_2^2(\Omega l_2)^{1/2}}\cos(2\Omega l_2 t + \tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}},$$

$$w \sim \frac{V}{4(ar)^{1/2}(a^2r^2+az)^{1/4}}\left[\frac{1-l_1^2}{l_1(\Omega l_1)^{1/2}}\sin(2\Omega l_1 t - \tfrac{1}{4}\pi) + \right.$$
$$\left. + \frac{1-l_2^2}{l_2(\Omega l_2)^{1/2}}\sin(2\Omega l_2 t + \tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}}.$$

Thus the only significant difference here is that the amplitude of the oscillations decreases to zero, so that the ultimate motion is in general steady and two-dimensional and the axial velocity of the fluid is ultimately the same as that of the paraboloid.

In view of the ultimate singularity in the velocity gradients on the axis and body it seems that the detailed form (but probably not the general nature) of the solution is of doubtful validity in this neighbourhood.

In conclusion, I wish to thank Professor B. R. Seth for his kind guidance throughout the preparation of this paper.

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ROTATIONAL FLOW OF A LIQUID PAST A REGULAR POLYGONAL CYLINDER

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ROTATIONAL FLOW OF A LIQUID PAST A REGULAR POLYGONAL CYLINDER

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1. INTRODUCTION

THE problem of rotational flow of a liquid past cylinders has been considered by many writers. A. R. Mitchell[1] has investigated in detail the flow past circular, elliptic and parabolic cylinders. By an extension of Blasius' theorem for rotational flow, De[3] calculated the thrust due to the flow past cylinders whose cross-sections are bounded by certain family of curves.

In this paper we have considered the rotational flow past a cylinder of regular polygonal cross-section. The problem is simplified by transforming the area outside the polygon into the interior of a unit circle. We have also calculated the thrust on the cylinder due to liquid motion and it is found that the thrust is independent of the number of sides of the polygon provided the length of a side is taken as in (3.4). Further, the case of hypotrochoidal cylinders is considered and the thrust is calculated in each case. It may be noted that though the hypotrochoids belong to the same family of curves considered by De, these curves were excluded from his investigation, and his transformation also was different.

2. Let the undisturbed motion of the fluid consist of uniform velocity $-$ U along the x-axis and uniform vorticity $-\omega$, so that the stream function ψ_2 of the undisturbed flow is given by

$$\psi_2 = Uy - \tfrac{1}{2}\omega y^2.$$

The stream function ψ_1 for the disturbed flow satisfies the equation

$$\nabla^2\psi_1 + \omega = 0.$$

Putting $\psi = \psi_1 - \psi_2$, we have

$$\nabla^2\psi = 0.$$

The boundary conditions are that $\psi \to 0$ at infinity and

$$\psi = -Uy + \tfrac{1}{2}\omega y^2,$$

over the boundary.

3. *Conformal transformation.*—Taking the centre of the polygon as origin in the z-plane, the area outside the polygon can be transformed into the interior of a unit circle in the t-plane by means of the relation[2]

$$\frac{dz}{dt} = A \frac{\prod_{r=1}^{n} (t - t_r)^{2/n}}{t^2}, \tag{3.1}$$

where $t_1, t_2, \ldots t_n$ correspond to the n vertices of the polygon and lie on the circle $|t| = 1$. The symmetry of the figure shows that we can take the t's to be the n roots of $t^n = 1$. Thus (3.1) takes the simple form

$$\frac{dz}{dt} = A \frac{(1 - t^n)^{2/n}}{t^2}. \tag{3.2}$$

By adjusting the constant A, we can write

$$\begin{aligned} z &= \frac{1}{t}(1 - t^n)^{2/n} + 2 \int_0^t t^{n-2} (1 - t^n)^{2/n-1}\, dt \\ &= \frac{1}{t} + \sum_{r=1}^{\infty} a_{rn-1} t^{rn-1}, \end{aligned} \tag{3.3}$$

where

$$a_{n-1} = \frac{2}{n(n-1)},$$

$$a_{rn-1} = \frac{2}{n}\left(1 - \frac{2}{n}\right)\left(2 - \frac{2}{n}\right) \ldots \left(r - 1 - \frac{2}{n}\right) \frac{1}{\underline{|r}\,(rn - 1)}.$$

The infinite series on the right of (3.3) is absolutely convergent for $|t| \leqslant 1$. The radius of the circumscribed circle is given by

$$d = 1 + \sum_{r=1}^{\infty} a_{rn-1}, \tag{3.4}$$

and a side of the polygon is

$$2d \sin \frac{\pi}{n}.$$

4. On the boundary the stream function ψ satisfies the equation

$$\begin{aligned} \psi &= -Uy + \tfrac{1}{2}\omega y^2 + c \\ &= -\frac{U}{2i}(z - \bar{z}) - \tfrac{1}{8}\omega (z - \bar{z})^2 + c, \end{aligned} \tag{4.1}$$

where $\bar{z}$ is the complex conjugate of z.

From (3.2)

$$z = \sum_{r=0}^{\infty} a_{rn-1} t^{rn-1} \qquad (a_{-1} = 1),$$

$$(z - \bar{z})^2 = \left(\frac{1}{t} - \frac{1}{\bar{t}}\right)^2 + \sum_{r=1}^{\infty} a^2_{rn-1} (t^{2rn-2} + \bar{t}^{2rn-2} - 2)$$

$$+ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{rn-1} a_{sn-1} \{t^{rn+sn-2} + \bar{t}^{rn+sn-2}$$

$$- (t^{rn-sn} + \bar{t}^{rn-sn})\}, \quad r \neq s, \tag{4.2}$$

where $\bar{t}$ is the complex conjugate of t.

Now

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{rn-1} a_{sn-1} t^{rn-sn} = \sum_{m=1}^{\infty} K_m (t^{nm} + t^{-nm}), \tag{4.3}$$

where

$$K_m = \sum_{r=0}^{\infty} a_{rn-1} a_{rn+mn-1}. \tag{4.4}$$

Substituting in (4.1) we get

$$\psi = -\frac{U}{2i}\left[\left(\frac{1}{t} - \frac{1}{\bar{t}}\right) + \sum_{r=1}^{\infty} a_{rn-1} (t^{rn-1} - \bar{t}^{rn-1})\right]$$

$$- \tfrac{1}{8}\omega \left[\left(\frac{1}{t} - \frac{1}{\bar{t}}\right)^2 + \sum_{r=1}^{\infty} a^2_{rn-1} (t^{2rn-2} + \bar{t}^{2rn-2})\right.$$

$$+ \Sigma\, \Sigma\, a_{rn-1} a_{sn-1} (t^{rn+sn-2} + \bar{t}^{rn+sn-2})$$

$$\left. - 2 \sum_{m=1}^{\infty} K_m (t^{nm} + \bar{t}^{nm})\right] + c. \tag{4.5}$$

Taking R, θ as the polar co-ordinates in the t-plane, we can write on the boundary of the unit circle $R = 1$,

$$\psi = - U\left[- \sin\theta + \sum_{r=1}^{\infty} a_{rn-1} \sin (rn - 1)\,\theta\right]$$

$$- \tfrac{1}{4}\omega \left[\cos 2\theta + \sum_{r=1}^{\infty} a^2_{rn-1} \cos (2rn - 2)\,\theta\right.$$

$$+ \Sigma\,\Sigma\, a_{rn-1}\, a_{sn-1} \cos (rn + sn - 2)\,\theta$$

$$\left. - 2 \sum_{m=1}^{\infty} K_m \cos nm\theta\right] + c. \tag{4.6}$$

Therefore we take

$$\phi + i\psi = U\left\{t - \sum_{r=1}^{\infty} a_{rn-1} t^{rn-1}\right\}.$$

$$- \tfrac{1}{4} i\omega \left\{t^2 + \sum_{r=1}^{\infty} a^2_{rn-1} t^{2rn-2}\right.$$

$$\left. + \Sigma\Sigma\, a_{rn-1} a_{sn-1} t^{rn+sn-2} - 2\Sigma\, K_m t^{nm}\right\}. \quad (4.7)$$

Since

$$z^2 = \frac{1}{t^2} + \sum_{r=1}^{\infty} a^2_{rn-1} t^{2rn-2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{rn-1} a_{sn-1} t^{rn+sn-2},$$

we can write

$$\phi + i\psi = U\left\{-z + \left(t + \frac{1}{t}\right)\right\}$$

$$- \tfrac{1}{4} i\omega \left\{z^2 + t^2 - \frac{1}{t^2} - 2\sum_{m=1}^{\infty} K_m t^{nm}\right\}. \quad (4.8)$$

Therefore

$$\psi = -Uy + U\left(R - \frac{1}{R}\right)\sin\theta$$

$$- \tfrac{1}{4}\omega \left\{x^2 - y^2 + \left(R^2 - \frac{1}{R^2}\right)\cos 2\theta\right.$$

$$\left. - 2\sum_{m=1}^{\infty} K_m R^{nm} \cos nm\theta\right\} \quad (4.9)$$

Since on the boundary $R = 1$ and

$$x^2 + y^2 = z\bar{z} = \Sigma\, a^2_{rn-1} + 2\sum_{m=1}^{\infty} K_m \cos nm\theta,$$

the boundary condition is satisfied.

For the flow past the cylinder we get the stream function

$$\psi_1 = U\left(R - \frac{1}{R}\right)\sin\theta - \tfrac{1}{4}\omega\left\{x^2 + y^2 + \left(R^2 - \frac{1}{R^2}\right)\cos 2\theta\right.$$

$$\left. - 2\sum_{m=1}^{\infty} K_m R^{nm}\cos nm\theta\right\}. \quad (4.10)$$

5. *Liquid pressure on the cylinder.*—Let X, Y and M be the componens along the axes and the moment about the origin of the pressure thrusts on

the cylinder. Then by an extension of Blasius' theorem for rotational flow, we have[4]

$$X - iY = \tfrac{1}{2}\, i\rho \int_c (u - iv)^2\, dz \tag{5.1}$$

and

$$M = \text{real part of } -\tfrac{1}{2}\rho \int_c z\,(u - iv)^2\, dz, \tag{5.2}$$

c being the boundary of the cylinder.

From (4.10) the stream function for the flow past the cylinder is the imaginary part of

$$U\left(t + \frac{1}{t}\right) - \tfrac{1}{4} i\omega \left\{x^2 + y^2 + t^2 - \frac{1}{t^2} - 2 \sum_{m=1}^{\infty} K_m t^{nm}\right\}.$$

The complex velocity at any point is therefore given by

$$u - iv = \tfrac{1}{2} i\omega \bar{z} - \left\{U\left(1 - \frac{1}{t^2}\right) - \tfrac{1}{2} i\omega \left(t + \frac{1}{t^3}\right.\right.$$
$$\left.\left. - n \Sigma K_m m t^{nm-1}\right)\right\} \frac{dt}{dz}. \tag{5.3}$$

On the boundary

$$u - iv = \tfrac{1}{2} i\omega \sum_{0}^{\infty} a_{rn-1} t^{-rn+1} - \left\{U\left(1 - \frac{1}{t^2}\right) - \tfrac{1}{2} i\omega \left(t + \frac{1}{t^3}\right.\right.$$
$$\left.\left. - n \sum_{1}^{\infty} K_m m t^{nm-1}\right)\right\} \frac{dt}{dz}. \tag{5.4}$$

If

$$G = \tfrac{1}{2} i\omega \sum_{0}^{\infty} a_{rn-1} t^{-rn+1}$$

$$H = U\left(1 - \frac{1}{t^2}\right) - \tfrac{1}{2} i\omega \left(t + \frac{1}{t^3} - n \Sigma K_m m t^{nm-1}\right),$$

$$X - iY = \tfrac{1}{2} i\rho \int \left(G^2 \frac{dz}{dt} + H^2 \frac{dt}{dz} - 2GH\right) dt \tag{5.5}$$

and

$$M = \text{real part of} - \tfrac{1}{2}\rho \int_c z\left(G^2 \frac{dz}{dt} + H^2 \frac{dt}{dz} - 2GH\right) dt. \tag{5.6}$$

Now

$$\int G^2 \frac{dz}{dt}\, dt = 0$$

and

$$\oint \mathrm{GH}\, dt = \pi\omega\,(\mathrm{U} + \tfrac{1}{6} i\omega) \qquad \text{when} \quad n = 3,$$

$$= \pi\omega \mathrm{U} \qquad \text{when} \quad n \geqslant 4.$$

We have

$$\frac{dz}{dt} = \frac{1}{t^2}\left\{-1 + \sum_{r=1}^{\infty} a_{rn-1}\,(rn-1)\,t^{rn}\right\}.$$

The series on the right is absolutely convergent for $|t| \leqslant 1$, and

$$\left|\sum_{1}^{\infty} a_{rn-1}\,(rn-1)\,t^{rn}\right| < \Sigma\,|a_{rn-1}\,(rn-1)|.$$

$$< 1.$$

Therefore

$$\frac{dt}{dz} = -t^2\Big[1 + \sum_{1}^{\infty} a_{rn-1}\,(rn-1)\,t^{rn}$$

$$+ \{\Sigma\, a_{rn-1}\,(rn-1)\,t^{rn}\}^2 + \ldots\Big].$$

So we find

$$\int \mathrm{H}^2 \frac{dt}{dz}\, dt = -2\pi \mathrm{U}\omega + \frac{\pi i}{3}\,\omega^2 \qquad \text{when} \quad n = 3,$$

$$= -2\pi\mathrm{U}\omega \qquad \text{when} \quad n \geqslant 4.$$

Therefore for all values of $n \geqslant 3$

$$\mathrm{X} - i\mathrm{Y} = -2\pi i\rho \mathrm{U}\omega$$

or

$$\mathrm{X} = 0, \quad \mathrm{Y} = 2\pi\rho\mathrm{U}\omega.$$

Thus we see that the force exerted is a lift force and is independent of the number of sides of the polygon.

The contribution to the real part from the first and third integrals in (5.6) is clearly zero. The contribution from the second integral is $2\pi \mathrm{U}\omega$ when $n = 3$ and it is zero when $n \geqslant 4$. Therefore the moment is $-\pi\rho\mathrm{U}\omega$ when $n = 3$, and it is zero when $n \geqslant 4$.

6. *Cylinder of hypotrochoidal cross-section.*—The transformation

$$z = \mu\left(\frac{1}{t} + mt^n\right), \quad \mu > 0, \quad 0 \leqslant m \leqslant \frac{1}{n}, \tag{6.1}$$

where n is a positive integer, transforms the outside of the hypotrochoid in the z-plane on to the interior of the unit circle in the t-plane. The parametric representation of the curve in the z-plane corresponding to the circle $|t| = 1$ in the t-plane is given by

$$x = \mu(\cos\phi + m\cos n\phi)$$
$$y = \mu(-\sin\phi + m\sin n\phi). \tag{6.2}$$

If $m = 0$ the curve is a circle; if $n = 1$ the curve is an ellipse. When $n = 1/m = 2$ and $n = 1/m = 3$, the corresponding curves have three and four cusps respectively and they resemble in shape a triangle and a square respectively.

Proceeding in the same way as before, we get the stream function for the flow past the cylinder to be

$$\psi = \mu U\left(R - \frac{1}{R}\right)\sin\theta - \tfrac{1}{4}\omega\left[x^2 + y^2 + \mu^2\left\{\left(R^2 - \frac{1}{R^2}\right)\right.\right.$$
$$\left.\left.\times\cos 2\theta - 2mR^{n+1}\cos(n+1)\theta\right\}\right]. \tag{6.3}$$

The thrust on the cylinder for $n \geqslant 2$ and $mn = 1$ may be calculated as before. We find when $n = 2$,

$$X = -\tfrac{1}{2}\pi\rho\mu(U^2 + \tfrac{1}{2}\omega^2\mu^2),$$
$$Y = \tfrac{3}{4}\pi\rho\mu^2 U\omega;$$

when $n = 3$, $X = 0$,

$$Y = \tfrac{1}{3}\pi\rho\mu^2 U\omega;$$

when $n = 4$,

$$X = \tfrac{1}{8}\pi\rho\mu^3\omega^2, \qquad Y = \tfrac{7}{8}\pi\rho\mu^2 U\omega;$$

when $n \geqslant 5$,

$$X = 0,$$
$$Y = \left(1 - \frac{1}{2n}\right)\pi\rho\mu^2 U\omega.$$

The moment is found to be $-5/4\,\pi\rho\mu^3 U\omega$ when $n = 2$ and it is zero when $n \geqslant 3$.

Adjusting μ such that the arc length between two consecutive cusps of the hypotrochoid may correspond to the length of a side of the regular polygon, the results are tabulated below.

Cross-section	*Drag*	*Lift*	*Moment*
Equilateral triangle ..	0	$2\pi\rho U\omega$	$-\pi\rho U\omega$
Three-cusped hypotrochoid	$-(0\cdot445\pi\rho U^2 + 0\cdot176\,\pi\rho\omega^2)$	$0\cdot593\,\pi\rho U\omega$	$-0\cdot879\,\pi\rho U\omega$
Square	0	$2\pi\rho U\omega$	0
Four-cusped hypotrochoid	0	$0\cdot239\,\pi\rho U\omega$	0

In conclusion I wish to thank Prof. B. R. Seth for his kind guidance throughout the preparation of this paper.

2. Summary

The stream function for the rotational flow of liquid past a regular polygonal cylinder of n sides is obtained and the thrust on the cylinder due to liquid motion is calculated. It is found that the thrust is independent of the number of sides of the polygon when the length of a side is taken as $2d \sin (\pi/n)$, d being the radius of the circumscribed circle. Further, the corresponding problem for hypotrochoidal cylinders is also considered and the thrust is calculated in each case. The results for cross-sections of equilateral triangle and square are compared with those of hypotrochoids of three and four cusps respectively.

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