Abstract

The universal adjacency matrix $U(H)$ of a finite, simple, and undirected graph H is a linear combination of the adjacency matrix $A(H)$, the degree diagonal matrix $D(H)$, the identity matrix I and the all-ones matrix J, that is $U(H) = \alpha A(H) + \beta D(H) + \gamma I + \eta J$ and $\alpha(\neq 0), \beta, \gamma, \eta \in \mathbb{R}$. From this matrix, one gets several matrices as special cases, namely, the adjacency, Seidel, Laplacian, signless Laplacian, normalized Laplacian, generalized adjacency, convex linear combination of the adjacency and degree diagonal matrices, and also the bivariate polynomial of H. Moreover, choosing appropriate values of the parameters α, β, γ and η, we get all the above mentioned matrices for the complement of graph H. In this thesis, our objective is to study the universal adjacency spectrum and its associated eigenvectors of some graphs obtained from algebraic structures like rings and groups. For any square matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{v}$, the pair $(\lambda, \mathbf{v})$ is called an eigenpair of the matrix. For an algebraic graph H considered here, we find some eigenpairs of $U(H)$ explicitly and show that the remaining eigenpairs can be obtained from a symmetric matrix. However, in some particular cases, we determine the full universal adjacency spectrum explicitly. It may be noted that neither the universal adjacency spectrum nor the eigenvectors of any kind were computed before for the algebraic graphs considered here.

The zero divisor graph $\Gamma(R)$ of a finite commutative ring R with unity is a simple undirected graph with the set of all non-zero zero divisors of R as vertices, and two distinct vertices x and y are adjacent if and only if $xy = 0$. We obtain some structural properties of $\Gamma(\mathbb{Z}_n)$ and then study the spectrum of $U(\Gamma(\mathbb{Z}_n))$ in terms of distinct proper divisors of n, where $\mathbb{Z}_n$ is the ring of integers modulo n. Further, if n is a prime power, say $n = p^k$, then we explore the characteristic polynomial of the symmetric matrix associated with $U(\Gamma(\mathbb{Z}_{p^k}))$. The looped zero divisor graph $\mathring{\Gamma}(R)$ of R is the same graph as $\Gamma(R)$ together with a self-loop at vertex x if and only if $x^2 = 0$. We extend the definition of universal adjacency matrix to a looped graph, and then study some structural properties of $\mathring{\Gamma}(R)$ and eigenpairs of $U(\mathring{\Gamma}(R))$, where R is an arbitrary finite commutative ring with unity. We also get some additional results on the spectrum of $U(\mathring{\Gamma}(\mathbb{Z}_n))$, and find all the eigenpairs of $U(\mathring{\Gamma}(\mathbb{Z}_{pq}))$ and $U(\mathring{\Gamma}(\mathbb{Z}_{p^2q}))$, where p and q are distinct primes. Moreover, we discuss some structural properties and the universal adjacency eigenpairs of the looped zero divisor graph on a reduced ring.

Next, we consider the cozero-divisor graph $\Gamma'(R)$ of a finite commutative ring R with unity, where $\Gamma'(R)$ is a simple undirected graph with the set of all non-zero nonunits of R as vertices, and two distinct vertices x and y are adjacent if and only if $x \notin yR$ and $y \notin xR$. We discuss some structural properties of $\Gamma'(R)$ and then obtain eigenpairs of $U(\Gamma'(R))$. We get additional information on the spectrum of $U(\Gamma'(R))$ when R is a reduced ring and $\mathbb{Z}_n$.

Finally, we consider the commuting graph $\Delta(G,\Omega)$ and non-commuting graph $\overline{\Delta(G,\Omega)}$ defined on a finite non-abelian group G, where $\Omega \subseteq G$. The commuting graph of G is a simple undirected graph with vertex set Ω, and two distinct vertices x and y are adjacent if and only if $xy = yx$. The complement of $\Delta(G,\Omega)$ is the non-commuting graph of the group G. We find the universal adjacency eigenpairs of the commuting and non-commuting graphs for an arbitrary AC-group G with $\Omega = G$ or $G - Z(G)$. Moreover, we determine the full spectrum of $U(\Delta(G,\Omega))$, when G is the quasi-dihedral group, group of order pq (for distinct primes $p,\ q$), Hanaki groups, generalized quaternion group, metacyclic group, dihedral group, generalized dihedral group, group of order p^3, $(p+2)$-centralizer groups for a prime p, groups with commuting probability $\mathcal{P}(G) \in \{\frac{1}{2}, \frac{11}{27}, \frac{2}{5}, \frac{5}{14}, \frac{p^2+p-1}{p^3}\}$, and also when $\frac{G}{Z(G)}$ is some particular kind of groups like the Suzuki group, dihedral group and non-cyclic group of order p^2.

Keywords: Eigenpairs; Spectrum; Characteristic polynomial; Universal adjacency matrix; H-join of graphs; Equivalence classes; Zero divisor graph; Looped zero divisor graph; Cozero-divisor graph; Commuting graph; Non-commuting graph; Finite commutative ring with unity; Non-abelian AC-group; Ring of integers modulo n; Reduced ring.