Abstract

Spectral graph theory is a branch of mathematics that studies properties of a graph through linear algebraic results of matrices associated with the graph. In this thesis, our main objective is to study the spectrum of distance and generalized distance matrices of some simple connected graphs. The distance matrix of a simple connected n-vertex graph G is an $n \times n$ symmetric matrix $D(G) = (d_{ij})$, where d_{ij} is the distance between i^{th} and j^{th} vertices in G. The distance Laplacian and distance signless Laplacian matrices of G are $\mathcal{L}(G) = Tr(G) - D(G)$ and $\mathcal{Q}(G) = Tr(G) + D(G)$ respectively, where $Tr(G)$ is the diagonal matrix of transmissions of vertices in G. We introduce the generalized distance matrix $D_{\alpha,\beta}(G)$ (of a connected graph G) as $D_{\alpha,\beta}(G) = \alpha D(G) + \beta Tr_G$, $\alpha \neq 0$, $\alpha, \beta \in \mathbb{R}$, to study distance, distance Laplacian, and distance signless Laplacian matrix in a unified way. A connected graph G is called a t-partitioned transmission regular graph if there exists a vertex partition $\{V_1, V_2, \ldots, V_t\}$ of G so that for $1 \leq i, j \leq t$, and $x \in V_i$, $\sum_{y \in V_j} d(x, y)$ is a constant. The advantage of a graph G being t-partitioned transmission regular is that one gets a matrix $Q(G)$ (quotient matrix) of order t so that all the eigenvalues of $Q(G)$ are eigenvalues of $D_{\alpha,\beta}(G)$, including the $D_{\alpha,\beta}$-spectral radius.

It is known that minimal (k, g)-cages are distance regular (DR) graphs, and their subdivisions are distance biregular (DBR) graphs. For minimal (k, g)-cages, we give a formula of distance spectral radius in terms of k and g, and also determine polynomials of degree $\lfloor \frac{g}{2} \rfloor$, which is the diameter of the graph. This polynomial gives all distance eigenvalues when the variable is substituted by adjacency eigenvalues. We show that a minimal (k, g)-cage of diameter d has $d + 1$ distinct distance eigenvalues, and this partially answers a problem posed by Atik and Panigrahi (2015). We prove that every DBR graph is a 2-partitioned transmission regular graph and then give a formula for its distance spectral radius. By this formula we obtain the distance spectral radius of subdivision of minimal (k, g)-cages. We also determine the full distance spectrum of subdivision of some minimal (k, g)-cages.

For positive integers $m_1, m_2, \ldots, m_h$, a generalized balanced tree $T(m_1, m_2, \ldots, m_h)$ is a rooted tree of height h such that every vertex of depth i has m_{i+1} children, $0 \leq i \leq h - 1$. Here we show that $T(m_1, m_2, \ldots, m_h)$ is an $(h + 1)$-partitioned transmission regular graph. We find the distance spectral radius of $T(m_1, m_2, \ldots, m_h)$ through a quotient matrix of its distance matrix. We obtain the characteristic polynomial of $D(T(m_1, m_2, \ldots, m_h))$ in terms of that of some smaller matrices and give an idea to

find the full distance spectrum.

For an n-vertex connected graph H, we introduce a graph operation called H-product. We prove that H-product of some graphs are n-partitioned transmission regular graphs. Then we study the distance and generalized distance spectra of some H-product graphs. As a result, we obtain some graphs which are answers to the problem "Determine the connected graphs with three distinct distance eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ such that $\lambda_2 = 0$.", asked by Koolen et al. (2016). Finally, we study the generalized distance spectrum of some more t-partitioned transmission regular graphs, namely, the spider graphs S_n^m, radar graphs R_n^m and tern graphs T_n^m. We also find generalized distance spectral radius and explore the distance and distance signless Laplacian spectra of some of these graphs. Moreover, we find the full generalized distance spectrum of S_n^2 and R_n^1 (wheel graph).

Keywords: Distance spectrum; Distance Laplacian spectrum; Distance signless Laplacian spectrum; Generalized distance spectrum; Transmission regular graph; t-partitioned transmission regular graph; Quotient matrix; Equatible partition; Distance regular graph; Distance biregular graph; Minimal (k, g)-cages; Generalized balanced trees; H-product graphs; Spider graph; Radar graph; Tern graph.