Abstract

This thesis examines the properties of complex unit gain graphs via the spectra of various classes of matrices associated with them. A complex unit gain graph ($\mathbb{T}$-gain graph), $\Phi = (G, \varphi)$ is a graph where the function φ assigns a unit complex number to each orientation of an edge of G, and its inverse is assigned to the opposite orientation. A $\mathbb{T}$-gain graph is balanced if the product of the edge gains of each oriented cycle (if any) is 1. The adjacency matrix $A(\Phi)$ of a $\mathbb{T}$-gain graph Φ is defined canonically. The adjacency spectrum of Φ is the spectrum of $A(\Phi)$.

The first chapter is introductory. It contains a general outlook of the thesis. We collect needed fundamental concepts and the background material and a brief overview of the literature to set the stage for the study conducted in the subsequent chapters.

In Chapter 2, we establish that a connected bipartite $\mathbb{T}$-gain graph has exactly one positive eigenvalue if and only if it is balanced. After that, we show that a $\mathbb{T}$-gain graph Φ is balanced if and only if it is cospectral with its underlying graph G. Further, we characterize $\mathbb{T}$-gain graphs for which the spectral radii of Φ and the underlying graph G coincide. We obtain a characterization for bipartite graphs in terms of $\mathbb{T}$-gain graphs. In addition, we study the combinatorial meaning of the coefficients of the characteristic polynomial and permanent polynomial of Φ.

In Chapter 3, we establish bounds for the spectral radius of $\mathbb{T}$-gain graphs and identify classes of graphs and gains for which the inequality is sharp. Let $\lambda_1(\Phi)$ and $\rho(\Phi)$ be the spectral radius and the largest eigenvalue of $A(\Phi)$, respectively. For the Hermitian adjacency matrix $H(X)$ of a mixed graph X, it is known that $\lambda_1(H(X)) \leq \rho(H(X)) \leq 3\lambda_1(H(X))$. One of the main objectives of this chapter is to extend this bound for the $\mathbb{T}$-gain graphs. We show that if the real parts of the gains Φ are nonnegative, then the above bounds hold. Also, we provide some structural conditions on the graphs for which the above bounds hold (for any gains). For any complex unit gain graph Φ, $\rho(\Phi) \leq \Delta$, where Δ is the largest vertex degree of G. For each $k \in \mathbb{N}$, we introduce a Hermitian matrix $H_k(X)$ called the k-generalized Hermitian adjacency matrix. Here, for each k, the gains are from the inverse closed set $\{1, e^{\pm\frac{i\pi}{k+1}}\}$. We characterize the structure of graph X for which $\rho(H_k(X)) = \Delta$ holds.

In Chapter 4, we focus on the bounds of energy of $\mathbb{T}$-gain graphs in terms of vertex cover number, largest vertex degree, smallest vertex degree, etc. For a simple graph G, it is known that $2\tau - 2c \leq \mathcal{E}(G) \leq 2\tau\sqrt{\Delta}$, where τ and c are the vertex cover number and the number of odd cycles of G, respectively. Many authors extended the result for mixed

graphs, oriented graphs, and other graphs. In this chapter, we extend the above inequality for $\mathbb{T}$-gain graphs. Also, we completely characterize complex unit gain graphs for which the bounds are sharp. The characterization of one side of the above equality completely solves an open problem, which is proposed for Hermitian adjacency matrices, in a more general setting viz., for the gain graphs. For any triangle-free connected $\mathbb{T}$-gain graph, we establish a bound for the energy in terms of the minimum vertex degree.

In Chapter 5, we propose two notions of gain distance matrices $\mathcal{D}^{\max}_{<}(\Phi)$ and $\mathcal{D}^{\min}_{<}(\Phi)$ of a $\mathbb{T}$-gain graph Φ, for any ordering '$<$' of the vertex set. We characterize the gain graphs for which the gain distance matrices are independent of the vertex ordering. We introduce the notion of positively weighted $\mathbb{T}$-gain graphs and establish an equivalent condition for the balance of a $\mathbb{T}$-gain graph. Acharya's and Stanić's spectral criteria for balance are deduced as a consequence. Furthermore, we obtain some spectral characterizations for the balance of a $\mathbb{T}$-gain graph in terms of the gain distance matrices. A $\mathbb{T}$-gain graph is distance compatible if $\mathcal{D}^{\max}_{<}(\Phi) = \mathcal{D}^{\min}_{<}(\Phi)$ holds for the standard ordering of the vertices. We characterize the distance compatible bipartite $\mathbb{T}$-gain graphs. Moreover, we characterize distance compatible 2-connected $\mathbb{T}$-gain graphs.

Keywords: Signed graphs; mixed graphs; complex unit gain graphs ($\mathbb{T}$-gain graphs); adjacency matrices; distance matrices; Hermitian adjacency matrices; graph energy; vertex cover number; vertex ordering; positive weighted $\mathbb{T}$-gain graphs.