Exact solutions to certain hyperbolic system of partial differential equations with source terms and their applications

Abstract

It may be noted that several real life applications can be modeled as nonlinear system of partial differential equations. This thesis deals with deriving exact solutions for one-dimensional quasilinear hyperbolic system of partial differential equations with source terms. These solutions are restrictive, admit certain special properties and can be classified accordingly. Further, the corresponding interaction of these solutions give insights for several applications. We consider certain systems like Euler equations of gasdynamics with large friction, pressureless Cargo-LeRoux model, Rate-type material and nonhomogeneous shallow water equations. While there are numerous methods to obtain solutions of nonlinear systems, we use Lie symmetry method and differential constraint method.

In this regard, for the one-dimensional Euler system that accounts for gravity together with large friction, we use Lie symmetry analysis and derive certain optimal classes of subalgebra. We apply the reduction procedure to reduce the Euler system to a system of ordinary differential equations in terms of a new similarity variable for each class of subalgebra leading to invariant solutions. The evolution of characteristic shock and its interaction with the weak discontinuity by using one of the invariant solutions is studied. The properties of reflected, transmitted waves and jump in the shock acceleration influenced by the incident wave are characterized. We then move on to discuss the wave interactions in pressureless Cargo-LeRoux model with flux perturbation. We prove the existence and uniqueness of the solution to the Riemann problem and construct estimates for the components of the solution. Relevant amplitudes are determined while presenting the interactions of weak discontinuity with contact discontinuity and shock waves. We discuss the interaction of weak shocks in detail. Some test cases are treated to understand the effect of initial data and strength of the shock on the transmitted, reflected amplitudes and jump in the acceleration of the shock.

We further focus on deriving exact solutions to generalized Riemann problem for nonhomogeneous quasilinear hyperbolic system of partial differential equations describing rate-type material which emerge while dealing with stress balance laws in solid mechanics. We derive consistency conditions and constraint equations for the governing equations with the help of differential constraint method. We characterize the solution for an arbitrary function as initial data which will help us to find an exact solution for generalized Riemann problem. We show some graphical results to discuss the rarefaction properties. Finally, we obtain exact solutions to nonhomogeneous shallow water equations with the generalized Riemann initial data. A special case of the obtained solution provides well known rarefaction wave to the homogeneous case of the governing equations. We construct a convenient example for the generalized Riemann problem and study the behavior of the solution profiles.

Keywords: Lie symmetry analysis; Optimal classification; Invariant solution; Euler system with large friction; Riemann problem; Elementary wave solution; Wave interactions; Weak discontinuity; Cargo-LeRoux model with flux perturbation; Generalized Riemann problem; Exact solutions; Differential constraint method; Nonhomogeneous shallow water equations; Rate-type material.