Chapter 1

Introduction and Literature Survey

1.1 Introduction

It has been observed that a large class of problems in science, engineering, economics, management, operations research etc. can be represented through the following optimization problem:

$$\min \quad f(x)$$

subject to

$$g(x) \leq 0,$$

$$h(x) = 0,$$

$$x \in X,$$

where f is a real valued function on $\mathbb{R}^n$ and g, h are mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. The function f is called the objective function, g is called an inequality constraint, h is called an equality constraint and $x \in X \subseteq \mathbb{R}^n$ is called an abstract constraint. The above problem as a whole is termed as a "Mathematical Programming Problem". The presence of x can be suppressed by taking X into whole of $\mathbb{R}^n$. If one of the functions involved in the above optimization problem is nonlinear, then it is called nonlinear mathematical programming problem. There are many real life applications of the mathematical programming problem, for example (see Bazaraa *et al.* (1993), Craven (1995) and Jahn (1996)).

1.1.1 Duality Principle in Mathematical Programming

A basic result in mathematical programming problems is the duality theory which asserts that, given a (primal) minimization problem the infimum value of the primal problem cannot be smaller than the supremum value of the associated (dual) maximum problem, and the optimal values of the primal, and dual problems are equal. Many of these duality principles are based on the geometric relation. It states that the shortest distance from a point to a convex set is equal to the maximum of the distances from the point to a hyperplane separating the point from the convex set. Hence, the original minimization over vectors can be converted to maximization over hyperplane. This result is more applicable in several computational applications, where the choice is often made so that evaluating the dual maximum is comparatively easier than solving a primal minimization problem. Over the years, many generalizations of this result to non differentiable convex problems (see Schechter (1979)), and differentiable non-convex problems (see Hanson (1981), Pini and Singh (1999), Mishra and Rueda (2002)) have appeared in the literature. Varaiya (1967), Ritter (1967), Chandra (1969), Behera *et al.* (2008) have discussed the concept of duality in Banach spaces and also established some duality results.

Unfortunately unlike the linear programming case, for the general nonlinear programs the dual of dual need not be the original primal. The concept of symmetric dual programs, in which the dual of the dual equals the primal, was introduced by Dorn (1960) for the nonlinear programming problems. Duality is not defined uniquely. This has led to the introduction of some other forms of dual such as the Mangasarian type, the Wolfe type and the Mond-Weir type duals. The beauty of Mond-Weir type dual is that the objective function is same as in the primal problem. Mond and Hanson (1968b) extended symmetric duality to variational problems, given continuous analog of the previous results.

Consider the following pair of optimization problems:

(P_1) $\min f(x)$ subject to $x \in S$,

(D_1) $\max \phi(u)$ subject to $u \in F$.

A point $x^* \in S$ is a local optimal solution of (P_1) if there exists $\delta > 0$ such that $f(x) \geq f(x^*)$ for all $x \in S$, $\|x - x^*\| < \delta$. If $f(x) \geq f(x^*)$ for all $x \in S$, then x^* is called a global optimal solution of (P_1).

Weak Duality Theorem: Weak duality theorem states that for all $x \in S$ and for all $u \in F$, $f(x) \geq \phi(u)$.

Zero Duality Gap: Zero duality gap says that if (P_1) reaches a minimum at $x = x^*$, then (D_1) reaches a maximum at $u = u^*$, satisfying $f(x^*) = \phi(u^*)$.

Converse Duality Theorem: Converse Duality Theorem says that if (D_1) reaches a maximum at $u = \bar{u}$, then (P_1) reaches a minimum at $x = \bar{x}$, satisfying $\phi(\bar{u}) = f(\bar{x})$.

Weak Duality implies that the objective value at a primal feasible solution is greater than or equal to the objective value at a dual feasible solution. This property allows the possibility of the duality gap between a minimum, $f(x^*)$ of (P_1) and a maximum, $\phi(u^*)$, for D_1 namely $f(x^*) > \phi(u^*)$. Zero duality gap is the statement that there is no duality gap. Converse duality is the related statement, that under suitable conditions the solution of the primal is also the solution of the dual and the objective values of the primal and dual are same.

If the programs (P_1) and (D_1) satisfy the properties of weak duality and zero duality gap, then (D_1) is called a strong dual of (P_1) (see Ben-Israel (1986), Craven (1988)). If weak duality and strong duality hold, then (P_1) is a converse dual of (D_1). If all the three properties, weak duality, zero duality gap and converse duality hold then, (D_1) is called a proper dual of (P_1). If the problem (P_1) has some special structure, then (D_1) may happen to be considerable simpler than (P_1). Moreover, since (given strong duality) (P_1) can be solved by solving (D_1) instead, algorithms for solving the dual problem can be used.

1.1.2 Second and Higher Order Duality

Consider the nonlinear programmnig problem:

$$\text{(P)} \ \min_{x} \quad f(x)$$

subject to

$$g(x) \leq 0, \tag{1.1}$$

where f and g are twice differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

The first order dual problem is:

$$\text{(FD)} \ \max_{u,y} \ f(u) + y^T g(u)$$

subject to

$$\nabla \left[f(u) + y^T g(u) \right] = 0, \tag{1.2}$$

$$y \geq 0, \tag{1.3}$$

where $y \in \mathbb{R}^m$. Mangasarian (1975) formulated the following second order dual by introducing an additional vector $p \in \mathbb{R}^n$.

$$\text{(SD)} \ \max_{u,y,p} \ f(u) + y^T g(u) - \frac{1}{2} p^T \nabla^2 \left[f(u) + y^T g(u) \right] p$$

subject to

$$\nabla \left[f(u) + y^T g(u) \right] + \nabla^2 \left[f(u) + y^T g(u) \right] p = 0, \tag{1.4}$$

$$y \geq 0. \tag{1.5}$$

Further, by introducing two differentiable functions $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, Mangasarian (1975) established the higher order dual

$$\text{(HD)} \;\; \max_{u,y,p} \quad f(u) + y^T g(u) + h(u,p) + y^T k(u,p)$$

subject to

$$\nabla_p h(u,p) + \nabla_p \big(y^T k(u,p)\big) = 0, \tag{1.6}$$

$$y \geq 0, \tag{1.7}$$

where $\nabla_p h(u,p)$ and $\nabla_p \big(y^T k(u,p)\big)$ denote the gradient of h and $y^T k$, respectively with respect to p is of dimension $n \times 1$.

The study of second and higher order duality is useful due to the computational advantage over first order duality as it gives bounds for the value of the objective function when approximations are used (see Mond (1974), Mangasarian (1975), Jeyakumar (1986), Mond and Zhang (1998), Mishra and Rueda (2000)). Motivated to the concept of second and higher order duality in nonlinear programming problems which was introduced by Mangasarian (1975), Hanson and Mond (1987) generalized the class of invex functions to a new class named Type I functions.

Here, we introduce second order V-$\rho - (\eta, \theta)$-Type I and higher order $\rho - (\eta, \theta)$-Type I functions and also state the weak duality theorems for both second and higher order duality cases.

Let K be the constraint set of (SD) given by (1.4) and (1.5), let $\eta(x,u)$, $\theta(x,u)$, $p(x,u)$ be the vector functions from $K \times K$ to $\mathbb{R}^n$ and $\rho \in \mathbb{R}$.

The objective function $f(x)$ is said to be a second order V-$\rho - (\eta, \theta)$-Type I objective function and $g_i(x),\; i = 1, 2, ..., m$ is said to be second order V-$\rho - (\eta, \theta)$-Type I constraint functions at $u \in K$ with respect to $\eta(x,u)$, $\theta(x,u)$ and $p(x,u)$ if for all $x \in K$,

$$\begin{aligned} f(x) - f(u) \geq{} & \eta(x,u)^T [\nabla f(u) + \nabla^2 f(u) p(x,u)] \\ & - \frac{1}{2} p(x,u)^T \nabla^2 f(u) p(x,u) + \rho \|\theta(x,u)\|^2, \end{aligned} \tag{1.8}$$

and

$$-g_i(u) \geq \eta(x,u)^T[\nabla g_i(u) + \nabla^2 g_i(u)p(x,u)] \\ -\frac{1}{2}p(x,u)^T\nabla^2 g_i(u)p(x,u) + \rho_i\|\theta(x,u)\|^2, i = 1,2,..,m. \quad (1.9)$$

Theorem 1.1.1 (Weak Duality) Let x and (u,y,p) be feasible solutions for (P) and (SD) respectively. Suppose that f is second order V-$\rho-(\eta,\theta)$-Type I function and $g_i^{,}s$ $(i = 1,2,...,m)$ are second order V-$\rho_i-(\eta,\theta)$-Type I functions defined over the constraint sets (P) and (SD), respectively and $\rho,\ \rho_i \in \mathbb{R}$ with $\rho + \rho_i \geq 0$. Then

infimum (P) $\geq$ supremum (SD).

Now, we define the higher order $\rho - (\eta,\theta)$-Type I functions for the objective and constraints functions of (HD). Let L be the constraint set of (HD) given by (1.6) and (1.7), let $\eta(x,u)$, $\theta(x,u)$, $p(x,u)$ be the vector functions from $L \times L$ to $\mathbb{R}^n$ and $\rho \in \mathbb{R}$. Let $h(u,p)$ and $k(u,p)$ be the vector functions from $L \times \mathbb{R}^n$ to $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

The function $f(x)$ is said to be a higher order $\rho - (\eta,\theta)$-Type I and the function $g_i(x),\ i = 1,2,...,m$ are said to be higher order $\rho - (\eta,\theta)$-Type I at $u \in L$ with respect to $\eta(x,u)$, $\theta(x,u)$, $p(x,u)$, $h(u,p)$ and $k(u,p)$ if for all $x \in L$, (for notational convenience take p and η in place of $p(x,u)$ and $\eta(x,u)$, respectively.)

$$f(x) - f(u) \geq \eta^T\nabla_p h(u,p) + h(u,p) - p^T\nabla_p h(u,p) + \rho\|\theta(x,u)\|^2, \quad (1.10)$$

and

$$-g_i(u) \geq \eta^T\nabla_p k_i(u,p) + k_i(u,p) - p^T\nabla_p k_i(u,p) + \rho_i\|\theta(x,u)\|^2, i = 1,2,..,m. \quad (1.11)$$

Theorem 1.1.2 (Weak Duality) Let x and (u,y,p) be feasible solutions for (P) and (HD) respectively. Suppose that f is higher order $\rho - (\eta,\theta)$-Type I function and $g_i^{,}s$ $(i = 1,2,...,m)$ are higher order $\rho_i - (\eta,\theta)$-Type I functions defined over the constraint sets (P) and (HD), respectively and $\rho,\ \rho_i \in \mathbb{R}$ with $\rho + \rho_i \geq 0$. Then

infimum (P) $\geq$ supremum (HD).

Now we consider some numerical examples, which show that the second order duality gives tighter bound than the first order dual and the higher order duality gives tighter bound than the first as well as second order dual.

Example 1.1.1 Consider the problem:

$$\min \quad f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1 \tag{1.12}$$

subject to

$$g_1(x_1, x_2, x_3) = x_1^2 - x_3 \leq 0, \tag{1.13}$$

$$g_2(x_1, x_2, x_3) = -x_1x_3 + x_2^2 \leq 0, \tag{1.14}$$

$$g_3(x_1, x_2, x_3) = -2x_2 + x_3^2 \leq 0. \tag{1.15}$$

We have

$$\nabla f = \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{pmatrix}, \qquad \nabla^2 f = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\nabla g_1 = \begin{pmatrix} 2x_1 \\ 0 \\ -1 \end{pmatrix}, \qquad \nabla^2 g_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\nabla g_2 = \begin{pmatrix} -x_3 \\ 2x_2 \\ -x_1 \end{pmatrix}, \qquad \nabla^2 g_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\nabla g_3 = \begin{pmatrix} 0 \\ -2 \\ 2x_3 \end{pmatrix}, \qquad \nabla^2 g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

From the inequalities (1.13)-(1.15) it is clear that x_1, x_2 and x_3 are nonnegative and the minimal value is 0 at the point $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$. Suppose it is unknown and we desire to examine the value of an approximate solution.

From Theorem 1.1.1, we see that the infimum value of the primal problem (P) can not be smaller than the supremum value of the associated second order dual problem (SD). To illustrate second order duality let us compare a lower bound to the minimal value given by by this approximation ($u_1 = 1, u_2 = 1, u_3 = 1$) in problem (SD) with a lower bound in (FD). The value of (P) at $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is 3, which is thus an upper bound to the true optimal value.

We show that the point $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is not a feasible point of (FD) and hence it does not provide a lower bound at $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Suppose $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is a feasible point of (FD). Thus it satisfies

$$\nabla \left[f(u) + y^T g(u) \right] = 0,$$

$$\text{and } y \geq 0.$$

That is,

$$2y_1 - y_2 + 2 = 0, \tag{1.16}$$

$$2y_2 - 2y_3 + 2 = 0, \tag{1.17}$$

$$-y_1 - y_2 + 2y_3 + 2 = 0, \tag{1.18}$$

$$y_1,\ y_2,\ y_3 \geq 0. \tag{1.19}$$

Adding (1.16), (1.17) and (1.18) we have $y_1 + 6 = 0$, which implies that $y_1 = -6$ which contradicts to (1.19) and hence $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is not feasible solution for (FD).

We now find expressions for $y = (y_1, y_2, y_3)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta = (\theta_1, \theta_2, \theta_3)$, $p = (p_1, p_2, p_3)$ and $\rho \in \mathbb{R}$ for which the functions f, g_1, g_2 and g_3 are second order V-$\rho - (\eta, \theta)$-Type I and satisfy the constraints of (SD). Although such expressions are not necessarily unique, here we assign arbitrary although it happens that we find a set with best possible.

Now we develop a relation between η, θ, p and ρ such that at $u_1 = 1, u_2 = 1, u_3 = 1$; $f,\ g_1,\ g_2,\ g_3$ are second order V-$\rho - (\eta, \theta)$-Type I.

(a) For $f(x)$:

$$f(x) - f(u) \geq \eta(x,u)^T[\nabla f(u) + \nabla^2 f(u)p(x,u)] - \frac{1}{2}p(x,u)^T\nabla^2 f(u)p(x,u) + \rho\|\theta(x,u)\|^2.$$

That is

$$x_1x_2 + x_2x_3 + x_3x_1 - 3 \geq \begin{pmatrix}\eta_1 & \eta_2 & \eta_3\end{pmatrix}\begin{pmatrix} p_2+p_3+2 \\ p_1+p_3+2 \\ p_1+p_2+2\end{pmatrix}$$
$$-\frac{1}{2}\begin{pmatrix}p_1 & p_2 & p_3\end{pmatrix}\begin{pmatrix} p_2+p_3 \\ p_1+p_3 \\ p_1+p_2\end{pmatrix} + \rho(\theta_1^2+\theta_2^2+\theta_3^2),$$

since $x_1, x_2, x_3 \geq 0$ we require at most the following inequality

$$-3 \geq (p_2+p_3+2)\eta_1 + (p_1+p_3+2)\eta_2 + (p_1+p_2+2)\eta_3$$
$$-p_1p_2 - p_1p_3 - p_2p_3 + \rho(\theta_1^2+\theta_2^2+\theta_3^2). \tag{1.20}$$

Similarly,

(b) for $g_1(x)$:

$$0 \geq 2\eta_1 + 2p_1\eta_1 - \eta_3 - p_1^2 + \rho_1(\theta_1^2+\theta_2^2+\theta_3^2), \tag{1.21}$$

(c) for $g_2(x)$:

$$0 \geq -\eta_1 + 2\eta_2 - \eta_3 - p_3\eta_1 + 2p_2\eta_2 - p_1\eta_3 + p_1p_3 - p_2^2 + \rho_2(\theta_1^2+\theta_2^2+\theta_3^2), \tag{1.22}$$

(d) for $g_3(x)$:

$$0 \geq -2\eta_2 + 2\eta_3 + 2p_3\eta_3 + 2p_3^2 + \rho_3(\theta_1^2+\theta_2^2+\theta_3^2). \tag{1.23}$$

Conditions for $f,\ g_1,\ g_2,\ g_3$ satisfy the constraints of (SD) at $u_1 = 1, u_2 = 1, u_3 = 1$.

$$\nabla\left[f(u) + y^Tg(u)\right] + \nabla^2\left[f(u) + y^Tg(u)\right]p = 0,$$

$$\text{and } y \geq 0.$$

That is,

$$(2+2p_1)y_1-(1+p_3)y_2+p_2+p_3+2=0, \tag{1.24}$$

$$(2+2p_2)y_2-2y_3+p_1+p_3+2=0, \tag{1.25}$$

$$-y_1-(1+p_1)y_2+(2+2p_3)y_3+p_1+p_2+2=0, \tag{1.26}$$

$$y_1,\ y_2,\ y_3\geq 0. \tag{1.27}$$

If we put $p_1=-1, p_2=-1$ and $p_3=-1$ in the constraints of (SD), we have

$$y_3=0,$$

$$y_1=0,$$

$$\text{and } y_1,\ y_2,\ y_3\geq 0,$$

so we obtain a set of feasible solution of (SD).

At $u_1=1, u_2=1, u_3=1$ and $p_1=-1, p_2=-1, p_3=-1$, the objective function in (SD) has the value

$$\begin{aligned} & f(u)+y^Tg(u)-\frac{1}{2}p^T\nabla^2\left[f(u)+y^Tg(u)\right]p \\ = & -y_1-y_3 \\ = & \ 0. \end{aligned}$$

The conditions for which $f,\ g_1,\ g_2$ and g_3 be of second order V-$\rho-(\eta,\theta)$-Type I at $u_1=1, u_2=1, u_3=1$, and $p_1=-1, p_2=-1, p_3=-1$ are:

(a) $-3\geq -3+\rho(\theta_1^2+\theta_2^2+\theta_3^2)$

(b) $0\geq -\eta_3-1+\rho_1(\theta_1^2+\theta_2^2+\theta_3^2)$

(c) $0\geq \rho_2(\theta_1^2+\theta_2^2+\theta_3^2)$

(d) $0\geq -2\eta_2+2+\rho_3(\theta_1^2+\theta_2^2+\theta_3^2)$, which are all satisfied if $\eta_2 = 1$, $\eta_3 = 1$, $\theta_1 = \frac{2}{3}$, $\theta_2 = \frac{1}{3}$, $\theta_3 = \frac{2}{3}$, $\rho = -\frac{1}{2}$, $\rho_1 = 2$, $\rho_2 = -\frac{1}{3}$, $\rho_3 = -\frac{5}{6}$ and there is no restriction on η_1.

As 0 is the optimal value of (P) and the objective value of (SD) at $u_1 = 1, u_2 = 1, u_3 = 1$ is also 0 for the values of $y = (y_1, y_2, y_3)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta = (\theta_1, \theta_2, \theta_3)$, $p = (p_1, p_2, p_3)$, ρ, ρ_1, ρ_2 and ρ_3 that we have obtained, these values are best possible by Theorem 1.1.1, though not necessarily unique. Of course in general problem we would not know the optimal value of the primal problem, and would not know if the values for y, η, θ, p, ρ are best possible, but any set of values satisfying the conditions imposed will give a lower bound for the optimal value of the primal problem.

Example 1.1.2 Consider the problem:

$$\min \quad f(x_1, x_2, x_3) = e^{x_1} + x_2^2 + x_3^2 \tag{1.28}$$

subject to

$$g_1(x_1, x_2, x_3) = x_1 - x_3^2 \leq 0, \tag{1.29}$$

$$g_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1 \leq 0, \tag{1.30}$$

$$g_3(x_1, x_2, x_3) = x_1^2x_2 + x_1x_3 - x_3^2 \leq 0. \tag{1.31}$$

We have

$$\nabla f = \begin{pmatrix} e^{x_1} \\ 2x_2 \\ 2x_3 \end{pmatrix}, \qquad \nabla^2 f = \begin{pmatrix} e^{x_1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\nabla g_1 = \begin{pmatrix} 1 \\ 0 \\ -2x_3 \end{pmatrix}, \qquad \nabla^2 g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\nabla g_2 = \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{pmatrix}, \qquad \nabla^2 g_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\nabla g_3 = \begin{pmatrix} 2x_1x_2 + x_3 \\ x_1^2 \\ x_1 - 2x_3 \end{pmatrix}, \qquad \nabla^2 g_3 = \begin{pmatrix} 2x_2 & 2x_1 & 1 \\ 2x_1 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

From (1.29) we have $x_1 \geq 0$ and the minimal value is clearly 0 at the point $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$. Suppose it is unknown and we desire to examine the value of an approximate solution.

To illustrate the higher order duality let us compare a lower bound to the minimal value given by the approximation $(u_1 = 0, u_2 = 1, u_3 = 1)$ in the problem (HD) with a lower bound in (FD). The value of (P) at $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ is 1, which is thus an upper bound to the true optimal value.

To adopt the similar approach of Example 1.1.1 one can easily see that the point $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ is not a feasible point of (FD) as well as (SD), whereas the the point $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ is a feasible point of (HD).

Even the functions $f,\ g_1,\ g_2,\ g_3$ are second order V-$\rho-(\eta,\theta)$-Type I and satisfy the constraints of (SD), and higher order $\rho-(\eta,\theta)$-Type I and satisfy the constraints of (HD), respectively at $u_1 = 0, u_2 = 1, u_3 = 1,\ p_1 = 0, p_2 = -1$ and $p_3 = -1$ with respect to same $\eta,\ \theta$ and p. The conditions for which the functions $f,\ g_1,\ g_2$ and g_3 are second order V-$\rho-(\eta,\theta)$-Type I and higher order $\rho-(\eta,\theta)$-Type I are as follows

(i) $0 \geq \eta_1 + \rho(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(ii) $0 \geq \eta_1 + \rho_1(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(iii) $0 \geq \eta_3 + \rho_2(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(iv) $0 \geq \rho_3(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(v) $-1 \geq 6\eta_3 + \rho(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(vi) $1 \geq \eta_2 - 2\eta_3 + \rho_1(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(vii) $-1 \geq 2\eta_2 + \rho_2(\theta_1^2 + \theta_2^2 + \theta_3^2)$

(viii) $1 \geq -\eta_2 - \eta_3 + \rho_3(\theta_1^2 + \theta_2^2 + \theta_3^2)$

which are all satisfied if $\eta_1 = -1,\ \eta_2 = -1,\ \eta_3 = -1,\ \theta_1^2 + \theta_2^2 + \theta_3^2 = 1,\ \rho = 1,\ \rho_1 = -1,\ \rho_2 = 1$ and $\rho_3 = -1$.

Now we consider the following functions to verify the feasibility of (HD) at the point

$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$.

$$h(u,p) = p_1 u_1^2 + p_2 u_1 u_2 - 3p_3^2 u_3^2 - 1, \tag{1.32}$$

$$k_1(u,p) = p_1 u_1 + p_2 u_2^2 - 2p_3 u_3, \tag{1.33}$$

$$k_2(u,p) = p_1 u_1^2 + p_2 u_2 + p_2 u_3 - p_3 u_1 u_3, \tag{1.34}$$

$$k_3(u,p) = p_1 u_1 - 2p_2 u_2^2 - p_3 u_3^2 + p_2 u_3. \tag{1.35}$$

For $p_1 = 0,\ p_2 = -1,\ p_3 = -1$, the constraints of the higher order dual (HD) at $u_1 = 0,\ u_2 = 1,\ u_3 = 1$ is given by

$$y_1 + 2y_2 - y_3 = 0,$$

$$2y_1 + y_3 - 6 = 0,$$

$$\text{and } y_1, y_2, y_3 \geq 0,$$

and the objective function is

$$y_3 - 2.$$

Therefore the point $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ is a feasible point of (HD) and one solution is $y_1 = \frac{3}{2},\ y_2 = \frac{3}{4},\ y_3 = 3$ and the value of the objective function is 1.

Since the optimal value of (P) is 1 and the objective value of (HD) at $u_1 = 0,\ u_2 = 1,\ u_3 = 1$ is also 1 for the values of $y = (y_1, y_2, y_3),\ \eta = (\eta_1, \eta_2, \eta_3),\ \theta = (\theta_1, \theta_2, \theta_3)$, $p = (p_1, p_2, P-3)$, $\rho,\ \rho_1,\ \rho_2$ and ρ_3 that we have obtained, these values are best possible by Theorem 1.1.2, though not necessarily unique.

In our examples fortunately we find best possible solutions in both second and higher order duality cases. Also we verify that the solution is not unique. In Example 1.1.1 both the objective and the constraints functions satisfy second order V-$\rho - (\eta, \theta)$-Type I conditions with same η, θ and p and it shows that the first order dual have no solution but the second order dual have solutions. In Example 1.1.2, the objective and the constraints functions satisfy simultaneously both second order V-$\rho-(\eta,\theta)$-Type I and higher order $\rho-(\eta,\theta)$-Type I conditions at the same point with same η, θ and p. And it shows that the first as well as the second order dual have no solution whereas the higher order dual have solutions.

1.1.3 Generalized Invexity

Convexity plays a key role in variational, control and mathematical programming problems. Many significant results have been derived under convexity assumptions. In the classical theory of optimization, the theorems on sufficient optimality conditions and duality are based on convexity assumptions, which are rather restrictive in applications. An important feature in the use of convexity is that local optimality implies global optimality. Another much used property of convex functions is that they are always bounded on one side by their tangent hyper planes at any point, which facilitates the use of linear bounds and approximations. But a number of problems arising in the real world are still non-convex in nature. Generally, it is not easy to handle such type of problems. Solutions of these problems are analytically challenging and practically useful. Though it is analytically tough to solve these general non-convex problems, yet many attempts are being made in this direction by many researchers, notable among them are Hanson, Craven, Mond, Kaul and Kaur, Zhang, Bector, Jeyakumar and Zalmai etc.

In this thesis, attempts are being made to extend the range of applications of some familiar results of convex programming to more general situations which involves differentiable functions. A differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is convex if and only if

$$f(x)-f(y)\geq(x-y)^T\nabla f(y),\ \forall x,y\in\mathbb{R}^n. \tag{1.36}$$

Hanson (1981) observed that term $(x-y)$ on the right hand side of the above inequality plays no role in the proof of the sufficiency of the Karush-Kuhn-Tucker (KKT) conditions.

This motivated Hanson (1981) to introduce a class of differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ which satisfy

$$f(x) - f(y) \geq \eta(x,y)^T \nabla f(y), \ \forall x, y \in \mathbb{R}^n \tag{1.37}$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function. If $\eta(x,y) = x - y$ then f becomes a convex function. Hanson (1981) also proved that the KKT conditions are sufficient if the objective and constraints functions satisfy the above inequality with the same $\eta(x,y)$. A differentiable function satisfying (1.37) was termed 'invex' by Craven (1981a) and showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minimum. Mond and Hanson (1984) extended the concept of invexity to polyhedral cones while Craven (1981b) defined it for more general cones and gave second order conditions for invexity.

The class of invex functions defined by Craven (1981a) is quite large due to the flexibility of the term $\eta(x,y)$. But that does not mean that all functions can be shown to be invex with some $\eta(x,y)$. For example, (see Ben-Israel and Mond (1986), Pini (1991)), the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3$ is not invex with any $\eta(x,y)$, since the invexity relations breaks down at $x = 0$. Craven (1996) defined invexity at a point; and noted that a function which is invex at a point with some $\eta(x,y)$, need not be invex over the whole domain.

After that a number of other generalizations of invexity such as pseudoinvexity and quasiinvexity have been appeared. Later, the concept of $\rho-(\eta,\theta)-$invexity was introduced by Zalmai (1990) as a generalization of invex functions. Nahak and Nanda (2005) studied the vector optimization problems under $\rho - (\eta,\theta)-$invexity assumptions.

Definition 1.1.1 A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $\rho - (\eta,\theta)-$invex at a point $u \in S \subseteq \mathbb{R}^n$ if there exist $\eta : S \times S \to \mathbb{R}^n$, $\theta : S \times S \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in S$,

$$f(x) - f(u) \geq \left\langle f'(u), \eta(x,u) \right\rangle + \rho \|\theta(x,u)\|^2.$$

Definition 1.1.2 A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)-$pseudoinvex at a point $u \in S \subseteq \mathbb{R}^n$ if there exist $\eta : S \times S \to \mathbb{R}^n$, $\theta : S \times S \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in S$,

$$\big\langle f'(u), \eta(x,u) \big\rangle + \rho \|\theta(x,u)\|^2 \geq 0 \Rightarrow f(x) \geq f(u).$$

Definition 1.1.3 A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)-$quasiinvex at a point $u \in S \subseteq \mathbb{R}^n$ if there exist $\eta : S \times S \to \mathbb{R}^n$, $\theta : S \times S \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in S$,

$$f(x) \leq f(u) \Rightarrow \big\langle f'(u), \eta(x,u) \big\rangle + \rho \|\theta(x,u)\|^2 \leq 0.$$

Remark 1.1.1 (i) for $\rho > 0$, we say f is "strongly invex",

(ii) for $\rho = 0$, we say f is "invex",

(iii) for $\rho < 0$, we say f is "weakly invex".

Also we see that the above function $f(x) = x^3$ which is not invex at any η will be $\rho-(\eta,\theta)-$Invexity for suitable η and θ. Hanson (1993) introduced the second order invexity and studied the some duality results. Again Hanson (FSU Technical Report Number M-883) showed that the second order duality give tighter bound than first order duality. Recently, Gulati and Gupta (2009) mentioned about the higher order invexity for a pair of symmetric mathematical programming problems.

Definition 1.1.4 A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be higher order invex at $u \in S \subseteq \mathbb{R}^n$ if there exist $\eta : S \times S \to \mathbb{R}^n$, $h : S \times \mathbb{R}^n \to \mathbb{R}$ and $\rho \in \mathbb{R}$ such that for all $(x,p) \in S \times \mathbb{R}^n$,

$$f(x) - f(u) \geq \eta(x,u)^T [\nabla_x f(u) + \nabla_p h(u,p)] + h(u,p) - p^T \nabla_p h(u,p).$$

In our work we introduce the concept of generalized higher order invexity which follows as:

Definition 1.1.5 A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be higher order $\rho - (\eta, \theta)-$invex at $u \in S \subseteq \mathbb{R}^n$ if there exist $\eta,\ \theta : S \times S \to \mathbb{R}^n$, $h : S \times \mathbb{R}^n \to \mathbb{R}$ and $\rho \in \mathbb{R}$ such that for all $(x, p) \in S \times \mathbb{R}^n$,

$$f(x) - f(u) \geq \eta(x, u)^T[\nabla_x f(u) + \nabla_p h(u, p)] + h(u, p) - p^T \nabla_p h(u, p) + \rho\|\theta(x, u)\|^2.$$

Remark 1.1.2 (i) If $\rho = 0$, then the above definitions reduce to higher order invexity with respect to η and h, for example see Gulati and Gupta (2009).

(ii) When $h(u, p) = 0$ and $\rho = 0$, then higher order $\rho - (\eta, \theta)-$invexity reduces to invexity with respect to η (see Chandra and Kumar (1998)).

(iii) If $h(u, p) = \frac{1}{2} p^T \nabla_{xx} f(u) p$ and $\rho = 0$, the above higher order $\rho - (\eta, \theta)-$invexity reduces to second order invexity defined by Gulati *et al.* (2001) and Suneja *et al.* (2003).

Many authors have worked on symmetric, variational and control problems under generalized invexity assumptions. For example see (Mond and Hanson (1967), Chen (1996), Chen (2000), Nahak and Nanda (2000) etc.).

1.2 A Literature Review on Duality Theory of Symmetric Mathematical Programming Problems, Variational problems and Control Problems

In this section we describe some earlier results on symmetric programming problems, variational problems and control problems.

1.2.1 Symmetric Duality

In mathematical programming, a pair of primal and dual problems are called symmetric if the dual of dual is the primal problem; that is, if the dual problem is expressed in the form of the primal problem, then its dual is the original primal problem. The symmetric dual programs have applications in the theory of games. However, the majority of the dual formulations in the nonlinear programming do not possess this property. The first symmetric dual formulation for quadratic programs was proposed by Dorn (1960). Subsequently Dantzig *et al.* (1965) and Mond (1965) developed significantly the notion of symmetric duality and established the duality results for convex and concave functions with non-negative orthant as the cone. The same result has been generalized by Bazarra and Goode (1973) to arbitrary cones. Later Mond and Weir (1991) weakened the condition to pseudoconvexity and pseudoconcavity with non-negative orthant as the cone. Gulati *et al.* (1997) studied the symmetric duality for the variational problems. Nahak (1998a) considered a pair of multiobjective generalized symmetric dual nonlinear programming problems and established desired duality theorems under $\rho-(\eta,\theta)-$invexity assumptions. Mishra (2000) studied the second order symmetric duality under second order $F-$convexity/$F-$concavity and second order $F-$pseodoconvexity/$F-$pseudoconcavity for the second order Wolfe and Mond-Weir type model, respectively. Later Gulati *et al.* (2001) extended the results of Mishra (2000) and proved weak duality relations under η_1-convexity/η_2-concavity and η_1-pseudoconvexity/η_2-pseudoconcavity assumptions, respectively. They also established other duality relations by assuming the additional hypothesis of skew symmetry. Nahak and Nanda (2002) studied the symmetric duality results for a pair of multiobjective programming problems under pseudoinvexity assumptions. Suneja *et al.* (2003) considered a pair of multiobjective second order symmetric dual problems of Mond-Weir type and established weak, strong and converse duality results under the assumptions of $\eta-$bonvexity and $\eta-$pseudobonvexity. Khurana (2005) studied the symmetric duality in vector optimization problems involving generalized cone-invex functions. Mishra and Yang (2005) established the Wolfe and Mond-Weir type second order symmetric duality for the multiobjective mixed integer programs over arbitrary cones. By using the concepts of efficiency and second order invexity/second order pseudoinvexity, they established weak, strong and converse duality theorems for the corresponding dual models. Gulati *et al.* (2008) formulated Wolfe and Mond-Weir type second order symmetric duals and established appropriate duality results under $\eta-$bonvexity/$\eta-$pseudobonvexity assumptions.

Presently many researchers are working on higher order symmetric duality in mathematical programming (see Gulati and Gupta (2007), Yang *et al.* (2008) and Mohamed and El-Hady (2009)). Gulati and Gupta (2009) formulated the Wolfe and Mond-Weir type higher order symmetric duals and established appropriate duality results under higher order $\eta-$invexity/$\eta-$pseudoinvexity assumptions. Recently, Gulati and Geeta (2010a) extended the same results of Gulati *et al.* (2008) for the multiobjective programming problem under pseudoinvexity/$K-F-$convexity assumptions.

1.2.2 Variational Problem and Mathematical Programming

Calculus of variation is a powerful technique for the solutions of various problems appearing in dynamics of rigid bodies, optimization of orbits, theory of variations and many other fields. The subject whose importance is fast growing in science and engineering primarily concern with finding optimal value of a definite integral involving a certain function subject to fixed point boundary conditions. Courant and Hilbert (1943), quoting an earlier work of Friedrichs (1929), gave a dual relationship for a simple type of unconstrained variational problem. Subsequently, Hanson (1964) pointed out that some of the duality results of mathematical programming have analogues in variational calculus. Exploring this relationship between the mathematical programming and the classical calculus of variations, Mond and Hanson (1967) formulated a constrained variational problem as a mathematical programming problem and using Valentine's optimality conditions (Valentine (1937)) presented its Wolfe type dual variational problem under convexity. Later Bector *et al.* (1984) studied the Mond-Weir type duality for the problems of Mond and Hanson (1967) to weaken the convexity requirement into pseudoconvexity and quasiconvexity. After that Mond *et al.* (1988) introduced the notion of invexity in variational problems. Wolfe and Mond-Weir type duals for multiobjective variational problems were formulated by Nahak and Nanda (1996). They also proved weak and converse duality theorems under invexity assumptions and established a close relationship between these variational problems and nonlinear multiobjective programming problems. Again Nahak and Nanda (1997b) generalized the results of Nahak and Nanda (1996) under pseudoinvexity assumptions.

Mishra and Mukerjee (1994) generalized $(F, \rho)-$convexity introduced by Preda (1992) and based on it Ahmad and Gulati (2005) established the duality results in multiobjective variational problems. Nahak and Nanda (2000) studied the symmetric duality with pseudoinvexity in variational problems. Recently, Nahak and Nanda (2007) established optimality conditions and duality results for the multiobjective variational problems under $V-$invexity assumptions. As the concept of invexity has allowed the convexity requirements in a variety of mathematical programming problems to be weakened, Mond and Smart (1989) extended the duality results for a class of nondifferentiable variational problems, treated in Chandra *et al.* (1985) with Wolfe and Mond-Weir type duals.

Since mathematical programming and classical calculus of variations have undergone independent development, it felt that the mutual adaption of ideas and techniques may prove fruitful. Motivated by this thought, Husain and Jabeen (2005) studied a class of constrained variational problems involving higher order derivatives. They established many duality results for the Wolfe and Mond-Weir type duals under invexity and generalized invexity assumptions. Following the approach of Mangasarian (1975), Chen (2003) formulated the second order duality for the class of constrained variational problems

$$\min \int_a^b f\left(t, x(t), \dot{x}(t)\right) dt,$$

subject to

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0,$$

$$x(a) = 0, x(b) = 0;\ \dot{x}(a) = 0, \dot{x}(b) = 0,$$

and gave many second order duality results (weak, strong and converse duality) under invexity assumptions. Recently, Gulati and Geeta (2010b) studied the optimality conditions and the converse duality result for the second order multiobjective variational problems.

1.2.3 Control Theory and Mathematical Programming

The theory of optimal control attracted great attention soon after Pontryagin and his followers had published in the late 1950. Pontryagin and his students encountered some problems in engineering and economics that urgently needed solutions. They answered the questions, and in the process they identically introduced a new subject named as the calculus of variations.

Just what is control theory? Who or what is to be controlled and by whom or by what, and why is to be controlled? In a nutshell, control theory, sometimes called automation, cybernetics or system theory, is a branch of applied mathematics that deals with the analysis and design of machinery and other engineering systems so that these systems work, and work better than before.

The subject is important from four viewpoints: which are given below

(i) as an intellectual discipline within science and the philosophy of science;

(ii) as a part of engineering, with industrial applications;

(iii) as a part of the educational curriculum at university;

(iv) as a force in the world related to technological, economic and social problems of the present and the future.

The applications of control problems are very frequent, for example, to engineering problems, like the control design for autonomous vehicles or impulsive control problems (see Pereira (2001a), (2001b)), electrical power production (see Christensen (1987), Ringlee (1965)), economy (see Intriligator (1971)), medicine (see Swam (1984)), ecology (see Leitmann (1981)) and computer integrated manufacturing and Robotics (see Dakev (1995)), among others.

Both the mathematical programming and control theory can be naturally dichotomized into the study of properties of solution points, i.e., necessary and sufficient conditions, and the development and analysis of effective computational methods for obtaining a solution. Mathematical programming is recognized by its property of necessary conditions for optimality: the Kuhn-Tucker theorem, whereas control theory is recognized by Pontryagin's maximum principle. For the detail one can refer the very fundamental book written by Craven (1978).

Broadly, the control problem is to be chosen, under given conditions, a control vector $u(t)$, such that the state vector $x(t)$ is brought from some specified initial state $x(t_0) = x_0$ to some specific state $x(t_1) = x_1$ in such a way that the given functional minimized. A more precise mathematical formulation is given in problem (CP) below.

$$\text{(CP)}\ \min \int_{t_0}^{t_1} f\left(t, x(t), u(t)\right) dt, \tag{1.38}$$

subject to

$$G\left(t, x(t), u(t)\right) = \dot{x}(t), \tag{1.39}$$

$$H\left(t, x(t), u(t)\right) \leq 0, \tag{1.40}$$

$$x(t_0) = x_1, x(t_1) = x_2. \tag{1.41}$$

A number of duality theorems for the control problems appeared in the literature (see, for example, (1965), Kreindler (1966)). Mond and Hanson (1968a) established duality results for the above control problem (CP) under the convexity assumptions. Mond and Smart (1988) had proved the above problem (CP) under invexity assumptions and also shown that for invex functions, the necessary conditions for optimality in the control problem are also sufficient. Bhatia and Kumar (1995) formulated the multiobjective control problems with $\rho-$pseudoinvexity, $\rho-$strictly pseudoinvexity, $\rho-$quasiinvexity or $\rho-$strictly quasiinvexity. Wolfe and Mond-Weir type duals for multiobjective control problems was formulated by Nahak and Nanda (1998b). They also established weak, strong and converse duality results under pseudoinvexity/quasiinvexity assumptions. Zhian and Qingkai (2001) discussed the duality results for multiobjective control problems with the assumptions of generalized invexity defined by Mond and Smarts (1988). Nahak and Nanda (1997a) discussed the efficiency and duality for multiobjective variational control problems with $(F, \rho)-$convexity. Park and Jeong (2004) established the weak, strong and converse duality theorems for multiobjective fractional generalized control problems. Gulati *et al.* (2005) studied the optimality conditions and duality results for multiobjective control problems. Arana *et al.* (2009a) provided a class of functionals, called KT-invex and showed that Kuhn-Tucker points being an optimal solutions for the control problem. Again Arana *et al.* (2009b) extended this result for new weaker conditions on the involved functionals, which is FJ-invexity. Recently, Khazafi *et al.* (2010) introduced the classes of (B, ρ)-type I and generalized (B, ρ)-type I, and derived various sufficient optimality conditions and mixed type duality results for multiobjective control problems under (B, ρ)-type I and generalized (B, ρ)-type I assumptions.

1.3 Objectives and Scope of the Thesis

The main objective of the thesis is to establish various duality results of different types of optimization problems. There are many problems in real life situations for which the first order dual has no solution whereas the second order dual has solutions. Also it has been observed that the first as well as second order have no solution whereas the higher order dual may provide a solution. This motivates us to introduce the second and higher order duality of the general nonlinear programming problems. The second and higher order duality for the symmetric mathematical programming have been established under generalized invexity assumptions. The variational and control problems have been given special attention to the optimization theory which is concerned with problems involving infinite dimensional cases. Besides this, the optimization problems in general Banach space have also been considered.

Convexity assumptions makes the solution of an optimization problem relatively easy and assures global optimal results. But there are many optimization problems which contain nonconvex functions and functionals. Therefore, it is essential to generalize the notion of convexity and to explore the extent of the validity of results to larger classes of optimization problems.

1.4 Organization of the Thesis

The thesis is organized in nine chapters. The first chapter presents the introduction and literature survey on the related work. Chapter 2 to Chapter 8 deals with a relevant contribution of the author in the field of nonlinear optimization. The last chapter includes a brief conclusion and outlines of some unsolved problems for further investigations. The bibliography section and the research publications of the authors are presented at the end. The chapter wise summary of the proposed work is given below.

In **Chapter 2**, two pairs of second order symmetric dual programs such as Wolfe type and Mond-Weir type are considered and appropriate duality results are established. Second order $\rho-(\eta,\theta)-$bonvexity and $\rho-(\eta,\theta)-$boncavity of the kernel function are studied. It is also observed that for a particular kernel function, both these pairs of programs reduce to the general nonlinear problem introduced by Mangasarian. (A part of this Chapter will appear as one chapter of the book ***Recent Contributions on Nonconvex Optimization***, Springer Publ., 2010.)

In **Chapter 3**, under the higher order $\rho-(\eta,\theta)-$invexity assumptions, the weak, strong and converse duality results are established for the following pair of Wolfe type higher order symmetric dual problems.

Wolfe type primal problem (WP)

min $\left\{f(x,y)+h(x,y,p)-p^T\nabla_p h(x,y,p)-y^T\nabla_y f(x,y)-y^T\nabla_p h(x,y,p)\right\}$

subject to

$$\left[\nabla_y f(x,y)+\nabla_p h(x,y,p)\right]\in C_2^*,$$

$$x \in C_1,$$

Wolfe type dual problem (WD)

$$\max \left\{ f(u,v) + g(u,v,r) - r^T \nabla_r g(u,v,r) - u^T \nabla_x f(u,v) - u^T \nabla_r g(u,v,r) \right\}$$

subject to

$$\left[\nabla_x f(u,v) + \nabla_r g(u,v,r) \right] \in C_1^*$$

$$v \in C_2,$$

where C_1 and C_2 represent closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, with non-empty interiors. $f : S_1 \times S_2 \to \mathbb{R}$, $g : S_1 \times S_2 \times \mathbb{R}^n \to \mathbb{R}$ and $h : S_1 \times S_2 \times \mathbb{R}^m \to \mathbb{R}$ are real differentiable functions and $S_1 \subset \mathbb{R}^n$ and $S_2 \subset \mathbb{R}^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

To weaken the higher order $\rho-(\eta,\theta)-$invexity assumptions to higher order $\rho-(\eta,\theta)-$pseudoinvexity and higher order $\rho-(\eta,\theta)-$quasiinvexity, Mond-Weir type higher order duality has also been studied. It has been observed that several known results can be deduced as special cases. If $h(u,p) = \frac{1}{2}p^T \nabla_{xx} f(u)p$, the higher order $\rho-(\eta,\theta)-$invexity defined in **Chapter 3** reduces to second order generalized invexity of **Chapter 2**. Furthermore, **Chapter 2** is a particular case of **Chapter 3** for $h(x,y,p) = \frac{1}{2}p^T \nabla_{yy} f(x,y)p$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$.

The optimization problems discussed in the previous chapters are only for finite dimensional. However, a great deal of optimization theory is concerned with problems involving infinite dimensional cases. Two types of problems fitting into this scheme such as variational problems and control problems are considered in this thesis.

Chapter 4 describes the second order duality of the variational problem (VP).

$$\text{(VP)} \;\; \min \int_a^b f\left(t, x(t), \dot{x}(t)\right) dt,$$

subject to

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0,$$

$$x(a) = 0, x(b) = 0;\;\; \dot{x}(a) = 0, \dot{x}(b) = 0,$$

where f and g are twice continuously differentiable functions from $I \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively. Under the $\rho-(\eta,\theta)-$invexity assumptions on the functions involved, duality relations (weak, strong and converse duality) between (VP) and the corresponding Mangasarian and Mond-Weir type second order dual problems are established. (A part of this Chapter has appeared in ***Computers and Mathematics with Applications***, 60 (2010) 3072–3081.)

Chapter 5 discusses the higher order duality and higher order generalized invexity of the variational problem (VP). Higher order weak, strong and converse duality results for the Mangasarian and Mond-Weir type variational problems are studied under the generalized higher order invexity assumptions. For some particular values of the functions defined on **Chapter 5**, the higher order generalized invexity and duality reduce to the invexity and second order duality of **Chapter 4**.

Chapter 6 describes the $\rho-(\eta,\theta)-$invex and generalized $\rho-(\eta,\theta)-$invex functions for the control problem (CP).

$$\text{(CP)} \;\; \min \int_a^b f\left(t, x(t), \dot{x}(t), u(t)\right) dt,$$

subject to

$$g\left(t, x(t), \dot{x}(t), u(t)\right) \leq 0,$$

$$G\left(t, x(t), \dot{x}(t), u(t)\right) = 0,$$

$$x(a) = \gamma_1, x(b) = \gamma_2; \;\; u(a) = \delta_1, u(b) = \delta_2,$$

where f, g and G are twice continuously differentiable functions from $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}$, $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. Under the above generalized invexity assumptions the second order duality results (weak, strong and converse duality) of the Mangasarian and Mond-Weir type control problems are established. (A part of this Chapter has appeared in ***International Journal of Optimization: Theory, Methods and Applications***, 1 (2009) 302-317.)

In **Chapter 7**, the concept of higher order duality and higher order generalized invexity are introduced for the control problem (CP). Higher order weak, strong and converse duality theorems for the Mangasarian and Mond-Weir type control problems are studied under the generalized higher order invexity assumptions. It has been observed that the invexity and second order duality defined on **Chapter 6** is a special case of **Chapter 7**.

The optimization problems studied in the previous chapters are over $\mathbb{R}^n$, but **Chapter 4** to **Chapter 7** deals with infinite dimensional problems. **Chapter 8** deals with the notion of $\rho-(\eta,\theta)-$invex functions and generalized $\rho-(\eta,\theta)-$invex functions between Banach spaces. Mangasarian type second and higher order duality theorems are established under $\rho-(\eta,\theta)-$invexity assumptions. To more weaken the invexity requirements Mond-Weir type model is also discussed. (A part of this chapter will appear in ***Nonlinear Analysis: Hybrid Systems***(2010).)