Chapter 1

Introduction

In Operations Research (OR), a real life decision making problem is transformed into a mathematical model that attempts not only to explain the behavior of the system but also to find an optimal solution of that system. A Mathematical Programming (MP) problem is an optimization problem where we maximize/ minimize one or more mathematical function, known as objective function subject to certain constraints imposed on the problem, which are represented as mathematical equations or inequations with some restrictions on the decision variables. An MP problem is often extended over a parameter space p.

Mathematically, it can be represented as:

$$\max / \min : f(x; p) \tag{1.0.1}$$

subject to

$$g(x; p) \leq 0 \tag{1.0.2}$$

$$x \geq 0 \tag{1.0.3}$$

where $x \subset R^n$ and the domain of the functions f and g, which map $R^n :\rightarrow R$.

If we consider the linearity of the functions f and g, then the MP problem can be divided into two different problems, namely linear and nonlinear optimization.

Similarly, some known subsets of MP problem are called constrained and unconstrained optimization (depending on presence or absence of constraints), single objective and multi-objective optimization (depending on the number of objective functions), continuous, discrete, mixed integer optimization (depending on the nature of decision variables), deterministic, fuzzy, stochastic optimization (depending on the parameter space). It has been observed that the parameters which form parameter space may be multi-choice type i.e. there may exist a set of choices for a parameter, out of which only one is to be selected to optimize the objective function(s). Such type of MP problem is known as multi-choice optimization.

Multi-choice problems occur in real life at different situations, namely selecting a new car, selecting a new security personnel, implementing a new policy for a community etc.

1.1 Motivation

Healy Jr (1964) formulated a Linear Programming (LP) problem with some zero-one variables, which he named as multiple choice programming. Typically, he formulated a mixed integer programming, where the integer variables must be either zero or one, and the integer variables are divided into sets such that the sum of variables in each set is unity. Beale and Tomlin (1970) noticed that the multiple choice programming of non-convex optimization problems can be modeled with Special Ordered Sets (SOS) of variables, which requires at most one of the variables in each of these sets to be non-zero in the final solution. Since then multiple choice programming has received wider attention from numerous researchers in the fields of integer programming and combinatorics.

Two types of SOS have been noticed in literatures, namely $SOS1$ and $SOS2$. $SOS1$ are sets of non-negative variables where, for each set, at most one of the variables can be non-zero in the final solution. The most common application of $SOS1$ is multiple choice programming which can be found in the modeling of many integer programming problems in location, distribution, scheduling, etc.

$SOS2$ requires that, for each set, at most two of the variables can be non-zero in the final solution and must be adjacent. $SOS2$ has been widely used in separable programming to model nonlinear functions using sets of piece-wise linear functions.

A general multiple choice programming may be expressed mathematically as follows:

$$\min : Z = \sum_{j=1}^{n} c_j x_j \tag{1.1.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \ \ i \in M = \{1, 2, 3, \ldots, m\} \tag{1.1.2}$$

$$\sum_{j \in S_t} x_j = 1, t \in K = \{1, 2, ..., k\} \tag{1.1.3}$$

$$x_j \geq 0, \ \ j \in J = \{1, 2, 3, \ldots, n\} \tag{1.1.4}$$

$$x_j \in \{0, 1\}, \ \ j \in J' \subseteq J \tag{1.1.5}$$

where

$$N \supseteq S = S_1 \cup S_2 \cup S_3 \cup .... \cup S_k \tag{1.1.6}$$

$$S_p \cap S_q = \emptyset, \forall p, q \in K, p \neq q \tag{1.1.7}$$

The MP problem in equations (1.1.1)- (1.1.5) minimizes the objective function in (1.1.1) subject to m regular constraints (1.1.2) with the additional requirement to satisfy k multiple choice constraints (1.1.3). The non-negativity requirement (1.1.4) and the integrality requirement (1.1.5) divide the variables in MP problem into two major classes of variables, namely S and $N - S$. Those in S are binary variables, where S is composed by k mutually disjoint multiple choice sets, $S_1, S_2, \ldots, S_k$, indicated by equations (1.1.6) and (1.1.7). Those in $N - S$ are non-negative continuous variables. The constraint in equation (1.1.3) is referred as generalized upper bounding constraints.

Generalized upper bounding constraints are considered implicitly by Dantzig and Van Slyke (1967) to modify simplex method by manipulating a reduced basis (working basis) during simplex iterations in order to develop generalized upper bounding technique, which has proved to be an effective solution approach to an LP problem with generalized upper bounding constraints. A relation between 'generalized upper bounding technique' and 'collapsed spanning tree procedure' has been established by Robert and Terry (1983) not only to give a theoretical framework to the findings of Dantzig and Van Slyke (1967) but also to justify the manipulation of working basis during simplex iterations. Glover (1973) has studied the effect of convexity cuts in the solution space of a multiple choice programming problem theoretically to show that generated convexity cuts in multiple choice programming are capable of eliminating some solution points that are not removed by using the general integer programming techniques. Glover's study summarizes several theorems that are helpful in developing the practical solution approach to multiple choice programming with the multiple choice constraints and the zero-one integrality requirement considered implicitly.

Beale and Tomlin (1970) noticed that the special procedure may take better advantage of non-convex problems with 'special ordered sets of variables', that exist in the multiple choice programming problems during analysis of the branch-and-bound procedure by Beale and Small (1965) for non-convex optimization problems. Beale and Tomlin (1970) proposed a weighted mean method to identify a partitioning point for every SOS to formulate two sub-problems so that the problem can be solved using the branch-and-bound procedure. However, the performance of the weighted mean method depends on the proper weight assigned to each variable, usually through a user-defined reference row. Armstrong (1975) has considered the multiple choice programming problem with a maximization objective function. The regular constraints (1.1.2) have the strict equality instead of the inequality. Armstrong incorporates the partitioning strategy of weighted mean method into the branch-and-bound procedure to examine the effect of using various types of penalty criteria, such as the improved penalty (Armstrong and Sinha, 1974) and the different schemes of reordering the variables in $S_t, t \in K$ for branching. Like weighted mean method, network transformation method (Glover and Mulvey, 1982), reduced cost method (Martin Dennis et al., 1985), reformulation and transformation technique (Lin Dennis and Edward, 1991) have been studied to find exact solutions of multiple choice programming problems.

Some authors, namely Martin and Sweeney (1983); Bean (1984); Cooper and Farhangian (1984); Chang and Tcha (1985) have proposed heuristic algorithms in order to solve multiple choice programming. Some specific algorithms have also been developed to deal with specially structured multiple choice programming problems by (Johnson and Padberg, 1981; Dyer et al., 1984). However, the global optimum of general multiple choice programming are often found using a modified branch-and-bound procedure analogous to the one used in general mixed integer programming. The distinction of this modified branch-and-bound procedure lies in its partitioning (i.e., branching) strategy. Two partitioning strategies have been proposed in the literatures, namely the commonly used weighted mean method by Beale and Forrest (1976) and the reformulation-and-transformation technique by Lin Dennis and Edward (1991). The heuristic technique, which is studied by Martin and Sweeney (1983) is based on 'Ideal column' algorithm, where as 'Pivot and complement' algorithm is used by Chang and Tcha (1985) to solve multiple choice programming problem. Bean (1984) has proposed a 'Lagrangian relaxation' algorithm based on heuristic technique to solve the problem.

Let $C_t, t \in K$, is the set of cost coefficients associated with k mutually disjoint multiple choice sets, $S_1, S_2, \ldots, S_k$. If we deeply go through the multiple choice programming given in equations (1.1.1)- (1.1.5), we observe that the constraints (1.1.3) and (1.1.5) forces the MP problem to select exactly one coefficients from each $C_t, t \in K$. Hence the aim of the multiple choice programming is to select exactly one cost parameter from multiple choices to minimize the objective function. This motivates us to extend the concept to other parameters, namely aspiration level, resource usage per unit of activity, available resources etc. in an MP problem.

1.2 Literature Survey

The optimization problem, which is presented in (1.0.1)-(1.0.3) is well-defined only if the parameter space contains deterministic parameters. Parameters depend on data of real world applications. Uncertainty in data is inevitable in most of real world decision making problems. If parameter space p contains uncertain parameters, how small uncertain may be, we have to make sure that uncertainty in data is appropriately incorporated into the decision making process. This involves the selection of a suitable MP problem that fits uncertain parameters. Uncertainties are usually categorized either in randomness or vagueness, which lead to either Stochastic Programming (SP) or Fuzzy Programming (FP).

The selection of MP model that fits uncertain data normally depends on historical data. It has been realized that instead of real data the moments of the data are available. In this case the data are considered as random. Appropriate SP model is to be selected to deal with such data. An extensive study has been made by several researchers and practitioners on optimization under uncertainty after the initial investigation of Dantzig (1955). Stochastic optimization with chance constraints has been studied by Charnes and Cooper (1959). In order to study the optimization under uncertainty where discrete distributions are considered, Van Slyke and Wets (1969) suggested L-shaped decomposition algorithm, which was based on the idea of Benders (1962). Optimization under uncertainty can also be solved by sample average approximation method of Kleywegt et al. (2002).

SP with the complete knowledge of probability distribution are mentioned in most of the optimization problems under uncertainty. Such uncertain optimization considers two distinct stochastic programming problems, namely:

(i) Recourse programming/ two stage programming (Dantzig, 1955),

(ii) chance constrained programming/ probabilistic programming (Charnes and Cooper, 1959).

Charnes and Cooper (1959, 1963) have suggested a chance constrained programming technique which can be used to solve problems involving chance/ probabilistic constraints. They suggested three models with different objective functions are given below:

(a) the E-model which maximizes the expected value of the objective function,

(b) the V-model which minimizes the generalized mean square of the objective function,

(c) the P-model which maximizes the probability of the aspiration level, i.e. the goal of the objective function.

In the literature of SP various models have been suggested by several researchers, namely Infanger (1992); Kall et al. (1994); Birge and Louveaux (1997); Biswal et al. (1998); Sahoo and Biswal (2009); Ferstl and Weissensteiner (2010).

Sometime, the assumption of complete knowledge of probability distributions of random parameters does not hold in reality. Such type of uncertainty is known as distribution ambiguity. Minimax (worst-case) principle has been used by Zackova, (1966) to study distribution ambiguity. Several choices of modeling distribution ambiguity have been suggested by Dupacova, (2001). Solution procedure depends on the modeling choices. Shapiro and Kleywegt (2002) developed a sample average approximation method for a class of finite number of distributions, on the other hand Riis and Andersen (2005) applied cutting plane method for a class of discrete (scenario-based) distributions. The problem of moments is related to the characterization of a feasible sequences of moments. Necessary and sufficient conditions for sequences of moments with different settings are provided by *Schmdgen* (1991); Putinar (1993); Curto and Fialkow (1996). Moment results can also be used to solve optimizations over polynomial (Parrilo, 2000; Lasserre, 2001). Two different bibliography have been presented by Stancu-Minasian and Wets (1976) and *Dupacov* (2007).

Sometime, the historical data are not clearly available i.e. data are vague in nature. In this case Decision Maker (DM) selects FP method to fit the model. Ambiguity and vagueness are two type of uncertainties, which are treated in the FP. It can be classified into three categories, namely 'FP with vagueness', 'FP with ambiguity' and 'FP with vagueness and ambiguity' (Inuiguchi et al., 1994).

FP with vagueness represents the flexibility of the target values of objective functions as well as elasticity of the constraints. Therefore such type of FP is known as flexible programming. The second category in FP treats ambiguous coefficients of objective functions and constraints but does not treat relaxation in fuzzy goals and constraints. Linear programming with fuzzy coefficients has been investigated by several researcher, namely Tanaka and Asai (1984); Orlovski (1984). A remarkable development has been made by Dubois and Prade (1987). Since the fuzzy coefficients can be regarded as possibility distributions on coefficient values, therefore such type of FP is called as possibilistic programming. FP also takes care of ambiguous coefficients as well as vague preference of DM. It is called as fuzzy robust programming. A general MP problem with fuzzy coefficients and fuzzy preference relation has been proposed by Orlovsky (1980). Various FP techniques are discussed by Inuiguchi et al. (1992); Lai and Hwang (1992a). A bibliography has been presented in Wong and Lai (2010).

Nahmias (1978) introduced the concept of fuzzy variable as a possible theoretical framework from which a rigorous theory may ultimately be constructed about fuzziness. Zimmermann (1985) presented nonsymmetric flexible linear programming problems where the data are considered to be crisp but certain constraints are considered to be "fuzzy inequality" (Lai and Hwang, 1992b,a). It has explored the fuzzy variable linear programming problems. In his work he mentioned that under certain conditions the nonsymmetric flexible linear programming problem is equivalent to a fuzzy variable linear programming problem. Mahdavi-Amiri and Nasseri (2007) show that an fuzzy variable linear programming problem is the dual of a fuzzy number linear programming problem, in which the coefficients of the cost function are fuzzy. Therefore, methods for solving fuzzy variable linear programming problems can be used for solving both the nonsymmetric flexible linear programming and fuzzy number linear programming problems. Maleki et al. (2000); Maleki (2002) presented an auxiliary problem for solving a fuzzy variable linear programming problem, having only cost coefficients are fuzzy. Solution and duality of fuzzy variable linear programming is studied by Hashemi et al. (2006) where all parameters and variables are of fuzzy. Defuzzification methods also have been studied and applied to fuzzy control and fuzzy expert systems. The major idea behind these methods is to obtain a typical value from a given fuzzy set according to some specified characters (center, fuzziness, gravity, median, etc.) Lotfi et al. (2009) discussed fuzzy variable linear programming problems of which all parameters and variable are triangular fuzzy numbers. They use the concept of the symmetric triangular fuzzy number to defuzzify a general fuzzy quantity. They first approximated the fuzzy triangular number to its nearest symmetric triangular number, with the assumption that all decision variables are symmetric triangular type.

1.3 Multi-Objective Mathematical Programming

Many decision making problems have multiple objectives to be optimized simultaneously. These objectives cannot be optimized simultaneously since they are conflicting in nature. For example, a transportation problem may require the minimization of the total transportation cost and minimization of the total transportation time during transportation of goods. A production planning problem may require the meeting of demands while minimizing the use of a particular type of resource and maximizing profit. Mathematical formulation of a objective function and the constraints in an MP problem usually includes some parameters which are multi-choice in nature. In a resource allocation problem the parameters may represent economic values such as costs of various types of production, shipment costs, etc. When the constraints and its multiple objectives of an MP can be expressed as linear functions, we have a Multi-Objective Linear Programming (MOLP) problem. The basic approach to solve an MOLP problem is to determine a solution that represents an acceptable trade-off or compromise between objectives, or to determine a set of such solutions and allow the DM to choose a suitable solution among them. In this type of problem it is often necessary to replace the concept of "optimum" solution with that of "best compromise" solution. Several methods such as weighting method, utility function method, ranking or prioritizing method, efficient solution method have been developed by Zeleny (1976, 1982); Triantaphyllou (2000); Wallenius et al. (2008); Ho et al. (2010) to solve multi-objective decision making problems.

In a multi-objective programming problem, parameters such as coefficients and right-hand side target values of the constraints are assumed to be known as deterministic real numbers. However, in some real world decision making problems, we may find some situations where the expert knowledge is not so certain to specify the parameters as real numbers and cases where parameters fluctuate in certain interval. For example, demands of certain products, availability of resources/manpower cannot be estimated as a single real number with certainty. To cope with such uncertainties, stochastic programming approaches were proposed by Stancu-Minasian (1984); Slowinski and Teghem (1990). In stochastic programming approaches, we generally estimate proper probability distributions of parameters. However, the estimation is not always a simple task because of the following reasons: (1) historical data of some parameters cannot be obtained easily especially when we face a new uncertain variable and (2) subjective probabilities cannot be specified easily when many parameters exist. SP is an optimization method based on the probability theory, has been developed in various forms, namely two stage programming problem of Dantzig (1955), chance constrained programming of Charnes and Cooper (1959). Especially, for multi-objective stochastic linear programming problems, Stancu-Minasian (1984, 1990) has considered the minimum risk approach, while Leclercq (1982); Teghem Jr et al. (1986) have proposed interactive methods. For specific purposes, some diversified SP methods have been discussed in the literature by Abdelaziz et al. (2007); Poojari and Varghese (2008); Aouni and La Torre(2010).

On the other hand, we can estimate the possible ranges of the uncertain parameters. In such cases, one can represent the parameters by possible ranges of fuzzy sets. Fuzzy mathematical programming representing the uncertainty or ambiguity in decision making situations by fuzzy set concepts has attracted attention of many researchers, namely Lai and Hwang (1992b); Sakawa (1993); Rommelfanger (1996). Some of the developments in the area of FP are based on the seminal paper by Bellman and Zadeh (1970). The field has been enriched by the work of Zimmermann (1978, 2001). Three types of FP is generally considered, namely 'FP with vagueness', 'FP with ambiguity' and 'FP with vagueness and ambiguity'. FP with vagueness (flexible programming) deals with right hand side uncertainties while FP with ambiguity (possibilistic programming) takes care of uncertainties in all the involved parameters in an MP. In all types of FP, the membership function is used to represent the degree of satisfaction of constraints, the decision-maker's expectations about the objective function achievement level, and the range of uncertainty of coefficients. Fuzzy multi-objective linear programming, first proposed by Zimmermann (1978). Later various forms have been presented by several authors. For specific purposes, many diversified MP methods have been discussed in the literature by Parra et al. (1999); Lahdelma et al. (2005); Rommelfanger (2007).

Multi-Objective Transportation Problem (MOTP) is a special type of multi-objective decision making problem, that usually involves multiple, conflicting, and incommensurable objective functions. The transportation problem is one of the earliest application of linear programming problem. Hitchcock (1941) was the originator of transportation problem. Dantzig (1963), then followed by Charnes et al. (1953) presented efficient methods using the simplex algorithm in 1947. The balanced cost minimizing transportation problem is very widely studied and has many applications. Appa (1973) studied some useful variants of the cost minimizing transportation problem. The cost minimizing transportation problem with mixed constraints was discussed by Brigden (1974). He presented a study on a flow constrained cost minimizing transportation problem. Another important class of transportation problem is time minimizing transportation problem also called the bottleneck transportation problem. Perhaps, original contribution in the field of the time minimizing transportation problem is due to Hammer (1971). The transportation problem is also used to solve the time-cost minimization problem by Bhatia et al. (1976). Aneja and Nair (1979) presented a bi-criteria transportation problem. Lee and Moore (1973) studied the optimization of MOTP. Diaz (1979) and Isermann (1979) proposed the procedures to generate all non-dominated solutions to the multi-objective linear transportation problem. Intensive investigations on MOTP have been made by several researchers, namely Diaz (1978); Ringuest and Rinks (1987); Bit et al. (1992, 1993a); Prasad et al. (1993); Li et al. (2001); Liu and Zhang (2005); Gupta and Mehlawat (2007).

Transportation problem is a special type of LP problem stemmed from a network structure consisting of a finite number of nodes and arcs attached to them. Efficient algorithms have been developed for solving the transportation problem when the cost coefficients and the supply and demand quantities are known exactly. However, there are cases that these parameters may not be presented in a precise manner. For example, the unit shipping cost may vary in a time frame. The supplies and demands may be uncertain due to some uncontrollable factors. Uncertainty is due to involvement of random parameters and fuzzy parameters. However, in some cases it is believed that the demand in transportation problem may be multi-choice type. Algorithm for solving transportation problem with fuzzy constraints is introduced by ÓhÉigeartaigh (1982). Ammar and Youness (2005) investigated the efficient solution and stability of MOTP with fuzzy cost coefficients, supply, and demand quantities. Intensive investigations on MOTP with fuzzy parameters have been made by several researchers, namely Luhandjula (1987); Chanas and Kuchta (1996); Hussein (1998); El-Wahed (2001).

The stochastic transportation problem is used to determine the optimal quantities of a commodity to be shipped from a number of supply points to a number of demand points, at which the amount of goods demanded is uncertain, but described by stochastic variables with known distribution functions. The stochastic transportation problem was described by Elmaghraby (1960). Williams (1963) studied stochastic transportation model for petroleum transport as well proposed a cross decomposition algorithm to solve said problem. Holmberg and Joernsten (1984) compared several different methods based on decomposition techniques and linearization techniques for this problem. Intensive investigations on stochastic transportation problem with random parameters have been made by several researchers, namely Cooper and Leblanc (1977); Qi (1987); Chalam et al. (1994); Holmberg (1995).

If the data of the problem possess randomness then the MP model takes the form of SP model. Stochastic programming approaches can be used to solve such type of SP model. Similarly, if the data of the problem contains imperfect information as vagueness then FP model can be formulated. Fuzzy programming approaches can be used to solve such type model. Interval programming method takes care of some mathematical programming problems, which contains the data of finite interval in its parameter space. All the elements of the interval are considered when the problem is solved by mathematical programming techniques. In practice, all parameters may not be always continuous. Therefore some discrete choices can be considered for mathematical modeling. That is a parameter may take a set of discrete values. Such type of MP is named as Multi-Choice Programming (MCP). In the literature there exists a few research articles on MCP (Chang, 2007, 2008; Liao, 2009; Lee et al., 2009; Liao and Kao, 2010; Paksoy and Chang, 2010).

MCP is an MP model, in which DM is allowed to set multiple number of choices for a given parameter. Ravindran et al. (1987) have considered an MP model, where a choice can be made among at least two parameters, so that only one must hold. For example, there may be a choice out of N resources only one of them is to be selected optimally. For this purpose they have presented an equivalent MP model using N numbers of binary (0/1) variables.

Chang (2007) has presented an MP model, namely multi-choice goal programming for multi-objective decision making problems. Multi-choice goal programming allows the DM to assign multiple aspiration levels for a goal to avoid excess deviation from the decision.

The conceptual expression of multi-choice goal programming is as follows:

$$\min : \sum_{i=1}^{n} \omega_i \mid f_i(x) - g_i^{(1)} \ or \ g_i^{(2)} \ or \ \ldots \ g_i^{(k_i)} \mid \tag{1.3.1}$$

subject to

$$\sum_{i=1}^{n} \omega_i = 1 \tag{1.3.2}$$

$$\omega_i \geq 0, i = 1, 2, ..., n \tag{1.3.3}$$

$$x \in S \tag{1.3.4}$$

where ω_i is the weight attached to the i-th goal; $f_i(x)$ is a linear function of $x = (x_1, x_2, \ldots, x_n)^T$; $g_i^{(k_i)}$ is the k_i-th target value of i-th goal; S is feasible region and x is a decision vector.

Existing GP technique cannot be used to solve the multi-choice goal programming due to the presence of multiple aspiration levels. However, in order to deal with the multiple aspiration levels multiplicative terms of binary variables can be used in the model to find an equivalent MP model, which can be solved using standard mathematical programming techniques.

The use of multiplicative terms of binary variables is not a straight forward approach for implementation. It becomes a simple expression when the number of aspiration levels assigned to a goal is an integral power of 2. Further, Chang (2008) replaced multiplicative terms of the binary variables using continuous variable. Although the transformed MP model does not involve multiplicative terms of binary variables and correspond to a linear form of multi-choice goal programming which can be solved by linear programming method. However, it is unable to take care of all the choices of a parameter. In stead of considering discrete choices, the transformed MP model treats the multi-choice parameter as an interval by selecting minimum and maximum elements from the set as lower and upper limit of the interval, respectively. Some authors, namely Liao (2009); Lee et al. (2009); Liao and Kao (2010) have applied multi-choice goal programming approach of Chang (2007) to solve some real life decision making problems. Paksoy and Chang (2010) have applied revised multi-choice goal programming approach of Chang (2008) in order to deal with multi-choice parameters to solve a supply chain network design problem.

1.4 Solution Methodology

We present a Multi-Objective Mathematical Programming (MOMP) model as follows:

$$\max / \min : Z = f(x) \tag{1.4.1}$$

subject to

$$x \in S, \tag{1.4.2}$$

where Z=$[Z_1, Z_2, \cdots, Z_R]^T$ represents the objective function $f(x) = [f_1(x),\ f_2(x),\ \cdots,\ f_R(x)]^T$ and $R \geq 2$ is a vector valued function. The feasible region is the set

$$S = \{x \in R^n \mid h_i(x) \leq b_i, \forall i = 1, 2, 3, ..., m, x \geq 0\}.$$

It can also be expressed as:

$$S = \{x \in R^n \mid g_i(x) \leq 0, \forall i = 1, 2, 3, ..., m, x \geq 0\}$$

where $g_i(x)$= $h_i(x) - b_i$, $i = 1, 2, ..., m$.

MOMP problems arise in every branches of science, engineering, management science, and social science. Several methods have been suggested and a wide range of efficient algorithms have been employed to solve MOMP problems (Evans, 1984; Cohon, 2004; Wallenius et al., 2008; Przybylski et al., 2010).

1.4.1 Fuzzy Programming Method

Fuzzy Programming has been applied successfully in Multi-Criteria Decision Making (MCDM) problems and in MOMP problems as a tool. In MCDM problems the objective functions are represented by fuzzy sets and the decision set is defined as the intersection of all the fuzzy sets associated with the constraints. The decision rule is to select the solution having the highest membership value of the decision set. Zadeh (1965) first introduced the concept of fuzzy set theory. Later Zimmermann (1978) used fuzzy set theory concept with suitable choice of membership function and derived a fuzzy linear program, which is identical to present day maxi-min problem. He derived that the solutions obtained by fuzzy linear programming technique are always efficient. Biswal (1992) has presented fuzzy programming technique to solve a multi-objective geometric programming problem by introducing a new type of membership function. Bit et al. (1992, 1993b) have applied fuzzy programming technique to solve a multi-objective transportation problem. Fuzzy linear programming technique has been used in many areas in last three decades. Steps of the fuzzy programming technique are given below:

Step-1: Select the first objective function (i.e. $Z_r(x), r = 1$) and solve it as a single objective MP problem subject to the constraints. Let $x^{(1)}$ be the ideal solution. Then select the second objective function and find the ideal solution as $x^{(2)}$, continue the process R times for R different objective functions. Let $x^{(1)}, x^{(2)}, \dots, x^{(R)}$ be the ideal solutions for the respective objective functions.

Step-2: Evaluate all these objective functions at all these ideal solutions and formulate a pay-off matrix (Table 1.1) of size R by R as follows.

Table 1.1: Pay-Off Matrix

	$Z_1(x)$	$Z_2(x)$	$\dots$	$Z_R(x)$
$x^{(1)}$	Z_{11}	Z_{12}	$\dots$	Z_{1R}
$x^{(2)}$	Z_{21}	Z_{22}	$\dots$	Z_{2R}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x^{(R)}$	Z_{R1}	Z_{R2}	$\dots$	Z_{RR}

Step-3: From pay-off matrix (Table 1.1) determine the bounds for r-th objective function $Z_r(x)$, r=1,2,...,R. If an objective function is of maximization type find the best upper bound U_r^* and worst lower bound L_r^-. If an objective function is of minimization type find the best lower bound L_r^* and worst upper bound U_r^-, r=1,2,...,R.

Step-4: Associate a membership function $\mu_{Z_r}(x)$ to the r-th objective function $Z_r(x)$ as:

$$\mu_{Z_r}(x) = \begin{cases} 1, & if \;\; Z_r(x) \geq U_r^* \\ \frac{Z_r(x)-L_r^-}{U_r^*-L_r^-}, & if \;\; L_r^- < Z_r(x) < U_r^*, \quad r = 1, 2, 3, ..., R \\ 0, & if \;\; Z_r(x) \leq L_r^- \end{cases} \tag{1.4.3}$$

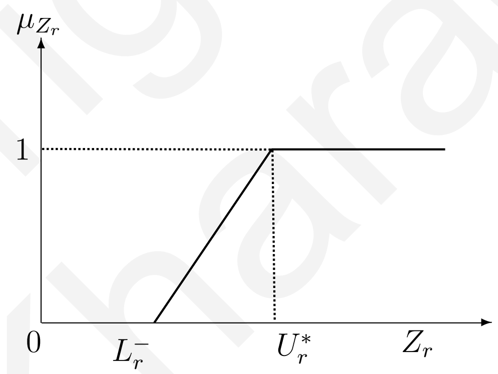

Figure 1.1: Membership Function of a Vector Maximization Problem

$$\mu_{Z_r}(x) = \begin{cases} 1, & if \;\; Z_r(x) \leq L_r^* \\ \frac{U_r^- - Z_r(x)}{U_r^- - L_r^*}, & if \;\; L_r^* < Z_r(x) < U_r^-, \quad r = 1, 2, 3, ..., R \\ 0, & if \;\; Z_r(x) \geq U_r^- \end{cases} \tag{1.4.4}$$

Step-5: **(a)** Use max-min operator with an augmented variable λ and formulate a single objective crisp MP problem as:

$$\max : \lambda \tag{1.4.5}$$

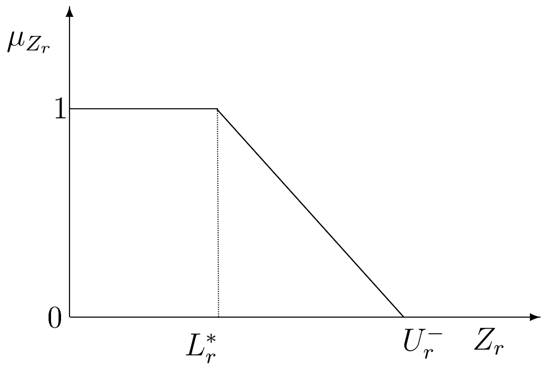

Figure 1.2: Membership Function of a Vector Minimization Problem

subject to

$$\lambda \leq \mu_{Z_r}(x),\ r = 1, 2, 3, \ldots, R \tag{1.4.6}$$

$$x \in S \tag{1.4.7}$$

where S is the feasible region of the MOMP model.

(b) Similarly, if we use min-max operator with an augmented variable λ, a single objective crisp MP problem can be formulated as:

$$\min : \lambda \tag{1.4.8}$$

subject to

$$\lambda \geq \mu_{Z_r}(x),\ r = 1, 2, 3, \ldots, R \tag{1.4.9}$$

$$x \in S \tag{1.4.10}$$

where S is the feasible region of the MOMP model.

Step-6: Solve the crisp model by using an appropriate mathematical programming method to find an optimal compromise solution x^*. Then evaluate all the objective functions at the optimal compromise solution x^*.

1.4.2 Goal Programming Method

Goal Programming (GP) is an important technique for DM to solve MCDM problems in finding a set of satisfying solutions. It was first introduced by Charnes and

Cooper (1957) and further developed by Lee (1972); Ignizio (1985a); Tamiz et al. (1998); Romero (2001). Ignizio (1985b); Romero et al. (1998); Biswal and Acharya (2008) established a framework that simplifies the problem for modified simplex method. At first, Ignizio developed under the name of MULTIPLEX, a unified structure encompassing the optimization methods such as: lexicographic vector minimization, Archimedean GP, lexicographic GP, and MINIMAX GP(Chebyshev GP).

The GP model is one in which all objectives are converted into goals. This conversion is accomplished by an '*aspiration level*' to each objective. Then one seeks the solution that minimizes the 'deviation' between that solution and the aspired solution. This fundamental idea leads to the following analytical framework. The general structure of the i-th goal can be expressed as:

$$h_i(x) + \eta_i - \rho_i = b_i, \quad i = 1, 2, \ldots, m,$$

where m : total number of goals,

$h_i(x)$: the mathematical expression for the i-th attribute,

b_i : the target values for the i-th goal,

η_i : the negative deviational variable, i.e., quantification of the under-achievement of i-th goal ($\eta_i \geq 0$),

ρ_i : the positive deviational variable, i.e., quantification of the over-achievement of i-th goal ($\rho_i \geq 0$).

After the formulation of all m number of goals, the next aim is to detect the unwanted deviational variables. The variables are unwanted in the sense that they are ones that DM wants to minimize. To illustrate this procedure the following cases are presented:

(i) Let's consider $h_i(x) \geq b_i$, as case-I where goal is attached to a minimization type objective. In this case the DM does not want under-achievement with respect to target b_i. Consequently, the unwanted deviational variable η_i to be minimized.

(ii) Let's consider $h_i(x) \leq b_i$, as case-II where goal is attached to a maximization type objective. In this case the DM does not want over-achievement with respect to target b_i. Consequently, the unwanted deviational variable ρ_i to be minimized.

(iii) Let's consider $h_i(x) = b_i$, as case-III where goal is to be achieved exactly. In this case the DM neither wants over-achievement nor under-achievement with respect to target b_i. Hence both the negative deviational variable η_i and positive one ρ_i are equally unwanted, making it necessary to minimize $\eta_i + \rho_i$.

The purpose of GP is to minimize the deviations between the achievement of goals and their aspiration levels. The minimization process can be accomplished with different GP variant. Basically, there are only three GP variants reported in literature. The most widely used is Lexicographic Goal Programming (LGP), which attaches preemptive priorities to the different goals in order to minimize the unwanted deviational variables in a lexicographic order. The second one is Weighted Goal Programming (WGP), which attempts to minimize a composite objective function formed by a weighted sum of unwanted deviational variables. The third is MINIMAX (Chebyshev) Goal Programming, which attempts to minimize the maximum deviation from stated goals.

We present a general goal programming model for the MOMP problem given in equations (1.4.1)-(1.4.2) as:

Find $\quad x = (x_1, x_2, \cdots, x_n)^T$, so as to

$$\begin{aligned} \textit{lexicographically} \ \min : \bar{a} &= \{a_1, a_2, ..., a_P\} \\ &= \{a_1(\eta,\rho), a_2(\eta,\rho), ..., a_P(\eta,\rho)\} \end{aligned} \tag{1.4.11}$$

Subject to

$$h_i(x) + \eta_i - \rho_i = b_i, \ \ i = 1, 2, 3, ..., m \tag{1.4.12}$$

$$f_r(x) + \eta_{r+m} - \rho_{r+m} = b_{m+r}, \ \ r = 1, 2, ..., R \tag{1.4.13}$$

$$\eta_i \ \geq \ 0, \ \ \rho_i \ \geq \ 0, \ \ i = 1, 2, ..., m + R \tag{1.4.14}$$

$$\eta_i.\rho_i = 0, \ \ i = 1, 2, ..., m + R \tag{1.4.15}$$

$$x \geq 0 \tag{1.4.16}$$

where $a_i(\eta, \rho)$, $i = 1, 2, ..., P$ are the linear functions of η, ρ, and P is the number of priorities in the achievement function $\bar{a}$.

1.4.3 Weighting Method

The weighting method is one of the basic method for solving multi-objective optimization problems. The idea of assigning weights to various objective functions, combining these into a single objective function, and parametrically varying weights to generate non-dominated solutions which has been proposed by Zadeh (2002). A basic assumption in the weighting method is that the objective functions are measured in the same unit. If the objective functions are not in the same unit, it can be transformed into the same unit before applying the method.

Mathematically, the weighting method can be stated as follows:

$$\max : Z = \sum_{r=1}^{R} w_r Z_r(x) \tag{1.4.17}$$

subject to

$$\sum_{r=1}^{R} w_r = 1, w_r \geq 0, r = 1, 2, 3, ..., R \tag{1.4.18}$$

$$x \in S \tag{1.4.19}$$

where S is the feasible region of the multi-objective MP problem. The coefficient w_r operating on the r-th objective function Z_r, is called a weight and can be interpreted as the relative weight of that objective function when compared to the other objective functions.

If the weights of various objective functions are interpreted as representing the relative preference of some DM, then the solution is equivalent to the best compromise solution i.e. the optimal solution relative to a particular preference structure. Also the optimal solution to the problem is a non-dominated solution provided all the weights are positive.

1.5 Preliminaries on Fuzzy Set Theory

Definition 1.5.1 (Fuzzy Subset) *Let X be a collection of distinct objects and x be an element of X. Then a fuzzy set $\tilde{A}$ in X is a set of order pairs*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where $\mu_{\tilde{A}}(x)$ is called the membership function or grade function of $\tilde{A} : X \to M$. Usually M is considered as the closed interval $[0, 1]$.

Definition 1.5.2 (r-level set) *The set of elements which belong to the fuzzy set $\tilde{A}$ at least to the degree r is called the r-level set. Mathematically, it is represented as:*

$$A_r = \{x \in X \mid \mu_{\tilde{A}}(x) \geq r\}$$

and

$$A'_r = \{x \in X \mid \mu_{\tilde{A}}(x) > r\}$$

is called a strong r-level set.

Definition 1.5.3 *A fuzzy number $\tilde{A}$ in parametric form is an interval in the form $[\underline{a}(r), \overline{a}(r)]$, where $\overline{a}(r)$ and $\underline{a}(r)$ are linear functions of r, $0 \leq r \leq 1$, which satisfies the following conditions:*

(i) $\underline{a}(r)$ *is a bounded, left continuous non-decreasing function over [0,1],*

(ii) $\overline{a}(r)$ *is a bounded, right continuous non-increasing function over [0,1],*

(iii) $\underline{a}(r)$ *and* $\overline{a}(r)$ *are right continuous at 0,*

(iv) $\underline{a}(r) \leq \overline{a}(r)$, $0 \leq r \leq 1$.

We denote this family of fuzzy numbers by $\mathcal{F}$.

A trapezoidal fuzzy number is represented as $\tilde{A}$=$(a^L, a^U, a^\alpha, a^\beta)$, where the interval $[a^L, a^U]$, a^α, a^β denote the core, left, and right spreads of trapezoidal fuzzy number, respectively.

The membership function of the trapezoidal fuzzy number is defined as:

$$\mu_{(\tilde{A})}(x) = \begin{cases} \frac{1}{a^\alpha}(x - a^L + a^\alpha), & a^L - a^\alpha \leq x \leq a^L \\ 1, & x \in [a^L, a^U] \\ \frac{1}{a^\beta}(a^U - x + a^\beta), & a^U \leq x \leq a^U + a^\beta \\ 0, & otherwise. \end{cases} \tag{1.5.1}$$

Its parametric form is given by

$$\underline{A}(r) = a^L - a^\alpha + ra^\alpha, \bar{A}(r) = a^U + a^\beta - ra^\beta.$$

Support function of $\tilde{A}$ is defined as follows:

$$Supp(A) = \overline{\{x|A(x) > 0\}},$$

where $\overline{\{x|A(x) > 0\}}$ is the closure of $\{x|A(x) > 0\}$.

The addition and scalar multiplication of fuzzy numbers are defined by the extension principle of Bellman and Zadeh (Bellman and Zadeh, 1970; Zimmermann, 1978.

Definition 1.5.4 *For arbitrary fuzzy numbers* $\tilde{A} = [\underline{a}(r), \overline{a}(r)]$ *and* $\tilde{B} = [\underline{b}(r), \overline{b}(r)]$, *the distance is calculated using the formula*

$$d(\tilde{A}, \tilde{B}) = \Big[\int_0^1 (\underline{a}(r) - \underline{b}(r))^2 dr + \int_0^1 (\bar{a}(r) - \bar{b}(r))^2 dr \Big]^{\frac{1}{2}} \tag{1.5.2}$$

1.5.1 Nearest Trapezoidal Defuzzification

Let $[\underline{b}(r), \overline{b}(r)]$ be the parametric form of fuzzy number $\tilde{B}$. To obtain a trapezoidal fuzzy number $\tilde{A}$=$(a^L, a^U, a^\alpha, a^\beta)$, which is the nearest to $\tilde{B}$ (not equal), we minimize the distance

$$\min : d(\tilde{A}, \tilde{B}) = \Big[\int_0^1 (\underline{a}(r) - \underline{b}(r))^2 dr + \int_0^1 (\bar{a}(r) - \bar{b}(r))^2 dr \Big]^{\frac{1}{2}} \tag{1.5.3}$$

$$= \Big[\int_0^1 ((a^L - a^\alpha + ra^\alpha) - \underline{b}(r))^2 dr + \int_0^1 ((a^U + a^\beta - ra^\beta) - \bar{b}(r))^2 dr \Big]^{\frac{1}{2}} \tag{1.5.4}$$

In order to minimize $d(\tilde{A}, \tilde{B})$, it is sufficient to minimize the function $D(\tilde{A}, \tilde{B}) = d^2(\tilde{A}, \tilde{B})$ (Saeidifar and Pasha, 2009).

$$\min : D(\tilde{A}, \tilde{B}) = \Big[\int_0^1 ((a^L - a^\alpha + ra^\alpha) - \underline{b}(r))^2 dr + \int_0^1 ((a^U + a^\beta - ra^\beta) - \bar{b}(r))^2 dr \Big] \tag{1.5.5}$$

We apply the necessary and sufficient conditions to find the minimum. Therefore we take the partial derivatives of the function in (1.5.5) with respect to left modal value a^L, right modal value a^U, left spread a^α and right spread a^β and equate them to zero. Thus we have four equations.

$$\frac{\partial D(\tilde{A}, \tilde{B})}{\partial a^L} = 2 \int_0^1 ((a^L - a^\alpha + ra^\alpha) - \underline{b}(r)) dr = 0 \tag{1.5.6}$$

$$\frac{\partial D(\tilde{A}, \tilde{B})}{\partial a^U} = 2 \int_0^1 ((a^U + a^\beta - ra^\beta) - \bar{b}(r)) dr = 0 \tag{1.5.7}$$

$$\frac{\partial D(\tilde{A}, \tilde{B})}{\partial a^\alpha} = 2 \int_0^1 ((a^L - a^\alpha + ra^\alpha) - \underline{b}(r))(r-1) dr = 0 \tag{1.5.8}$$

$$\frac{\partial D(\tilde{A},\tilde{B})}{\partial a^{\beta}} = 2\int_0^1 ((a^U + a^{\beta} - ra^{\beta}) - \bar{b}(r))(1-r)dr = 0 \tag{1.5.9}$$

Solving the equations for a^L, a^U, a^{α} and a^{β}, we obtain

$$a^L = \frac{a^{\alpha}}{2} + \int_0^1 \underline{b}(r)dr \tag{1.5.10}$$

$$a^U = -\frac{a^{\beta}}{2} + \int_0^1 \bar{b}(r)dr \tag{1.5.11}$$

$$a^{\alpha} = 6\int_0^1 \underline{b}(r)(2r-1)dr \tag{1.5.12}$$

$$a^{\beta} = 6\int_0^1 \bar{b}(r)(1-2r)dr \tag{1.5.13}$$

To check the sufficient condition we find the Hessian matrix of $D(\tilde{A},\tilde{B})$ as:

$$H = \begin{pmatrix} 2 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & \frac{1}{2} \\ -1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{pmatrix} \tag{1.5.14}$$

All the eigenvalues of the Hessian matrix H are computed as positive. So the Hessian matrix H is positive definite. Hence it satisfies the sufficient condition (Grzegorzewski and Mrówka, 2003).

1.6 Mathematical Tools and Techniques

The blitz of the computer revolution has given a great momentum to the development of computational techniques in OR. In order to deal with the complex real life decision making problems effectively, a huge amount of computation is required. To solve such complex problem a huge amount of time is needed. It may not be easy to check the correctness of the solution manually. The development of computers to perform arithmetic calculations thousands or even millions of times faster than a human being is a wonderful advantage in OR. In 1980, the development of increasingly powerful personal computers accompanied by good software packages for computing OR problems has attracted researchers and practitioners from interdisciplinary areas. Now-a-days OR softwares are frequently used by scientists and researchers to solve OR problems.

MP problems with multi-choice parameters are difficult to solve due to the multiple-choice component. There is no direct methodology to solve the problem. Some transformation techniques are presented to solve the problem. All the transformed models are mixed integer type NLP problem.

If the objective function and the constraints are convex functions, then the mixed integer NLP is known as convex mixed integer NLP, otherwise it is known as non-convex mixed integer NLP. The optimal solutions to non-convex mixed integer NLPs are local in nature, i.e. solutions are optimal within a restricted part of the feasible region (neighborhood), but it is not the entire feasible region. This does not happen when the objective function and the constraints are linear. The local optima are also the global optima in case of convex-mixed integer NLP i.e. solutions are optimal over the entire feasible region. In order to solve NLP, there is a need of calculating the first and second order derivatives of the nonlinear functions involved in the model. Therefore, the nonlinear functions are assumed to be smooth in optimization tools. Standard file format to express NLP/ mixed integer NLP problems is not available due to requirement of derivatives. The introduction of modeling languages provide a unique, intuitive, and flexible environment to express both linear and nonlinear models. These frameworks are interfaced to most of the modern solvers and provide a tool for computing the approximation of the derivatives required in the NLP/mixed integer NLP case.

Mixed integer NLP problems are NP-hard as they are the generalization of mixed integer (binary) linear programming problems, which are also NP-hard. Therefore mixed integer NLPs are very difficult to solve in practice. The presence of non-convexities in the MP model makes the problems more harder to solve. The availability of software for finding solutions to mixed integer NLP problems via the modeling language makes our computation easy. Since, GAMS/LINDOGLOBAL by Schrage (2008b) has the best score (79%) in the number of correctly claimed global optimal solutions to general nonlinear optimization problems with continuous as well as discrete variables, therefore in this thesis, GAMS/LINDOGLOBAL Schrage (2008b) solver and LINGO Schrage (2008a) software have been used to compute the MP models.

GAMS/LINDOGLOBAL supports most mathematical functions, including nonsmooth, discontinuous functions. The GAMS/LINDOGLOBAL optimization procedure employs branch-and-cut methods to break an NLP model into a list of subproblems. The GAMS/LINDOGLOBAL framework allows to implement both exact and heuristic methods in solution procedure. The GAMS/LINDOGLOBAL solver is accessible in four ways (Lin and Schrage, 2009) as follows:

(i) LINGO modeling language as the global solver option (Schrage, 2008b),

(ii) modeling system GAMS as the global solver option LINDOGLOBAL (Brooke et al., 2009),

(iii) spreadsheet optimizer as the global solver option (Savage, 1991),

(iv) LINDO API as a callable library (Lin and Schrage, 2009).

An optimal solution (probably global optimal) is obtained after a finite number of steps within given appropriate tolerances. One can use Alpha-Ecp (Westerlund and Pörn, 2002), BARON (69%) (Sahinidis and Tawarmalani, 2005), BONMIN (Lougee-Heimer, 2003), Cocos (76%) (Schichl et al., 2010), KNITRO (47%) (Byrd et al., 2006) etc. in GAMS platform to solve mixed integer NLPs. The test environment (Domes et al., 2010) can be referred for more details of updated tools.

1.7 Objectives and Scope of the Thesis

Objectives of the present work have been defined after an extensive literature survey based on state-of-the art problems of MP models and the parameter space associated with them. The main idea of defining the objectives for the present work lies not only in formulating different MP models, when the parameters are multi-choice type but also to propose solution methodologies for MCP problems. The main objectives are as follows:

(i) To study the solution procedure of Multi-Choice Linear Programming (MCLP) problems using some binary variables for particular cases (for limited number of choices) and then to extend it for a general case (for any finite number of choices),

(ii) to study the solution procedure of MCLP problems using some integer variables by formulating interpolating polynomials for the multiple choices,

(iii) to implement MCP methods in order to solve some probabilistic linear programming problems involving multi-choice parameters,

(iv) to implement MCP methods for fuzzy linear programming problems involving multi-choice parameters,

(v) to apply multi-choice programming to some real life decision making problems.

A linear programming problem has several parameters, namely cost coefficients, technical coefficients, resource limits (right hand side parameters of the constraints). In the thesis, the resource limits are considered as multi-choice type. MCP can be solved using genetic algorithm, neural network, transformation techniques etc. Transformation techniques have been used in order to formulate mathematical programming model for the multi-choice parameters. MCP can be applied to other MP problems, namely quadratic programming, geometric programming, bilevel programming, multilevel programming, and fractional programming problems. In this thesis deterministic, probabilistic, and fuzzy programming problems involving multi-choice parameters are discussed within the scope of the thesis.

1.8 The Structure of the Thesis

The research work under report and evaluation, has been organized into seven chapters. A brief outline of the same is described as follows:

Chapter 1 introduces a brief overview on the broad area of OR. Emphasis has been given on fuzzy programming, stochastic programming, multi-choice programming, which depends on the parameter space. A brief survey on fuzzy optimization, stochastic optimization, and MCP has been presented. It covers a review on theory, algorithm, and application. A brief review on tools and techniques commonly used in solving mixed integer Non-Linear Programming (NLP) problems are also presented. On the basis of the parameter space of MP models, the main objectives of the present work have been defined.

Chapter 2 considers MCLPs, where the right hand side parameters of some constraints are multi-choice type. Such problems cannot be solved by standard linear programming approaches. In order to solve such problem an equivalent MP model is formulated. This chapter covers a detailed description of transformation techniques of MCLPs with the help of some binary variables. It transforms an MCLP to a standard MP model, which can be solved by existing mathematical programming techniques. For each of the constraint there may exist multiple number of goals, out of which exactly one is to be selected. The selection of goals should be in such a manner that the combination of choices for each constraint should provide an optimal value of the objective function. There may be more than one combination which may provide an optimal solution. Binary variables are introduced in the transformation technique to formulate a mixed integer NLP model. Using standard nonlinear programming software optimal solution of the proposed model is obtained. For better demonstration of the methodology, it has been restricted to at best eight alternatives for a goal in the initial stage. Then, two different cases for a generalized transformation technique are presented to transform an MCLP problem to an equivalent MP model. Using any one of these transformation techniques the transformed model can be derived. Apart from single objective MCLP, we also consider a multi-objective MCLP problem, where some of the right hand side parameters of the constraints are multi-choice type. The selection of goals should be in such a manner that the combination of choices for each set should provide best compromise solution. Transformation techniques have been presented to obtain an equivalent multi-objective MP model, which is solved by fuzzy programming method. Each model is illustrated with a numerical example.

When we apply the transformation techniques presented in **Chapter 2** to formulate an equivalent MP model for an MCLP problem involving more number of choices for a parameter, the following difficulties are observed:

(i) selecting binary variables,

(ii) selecting bounds for the sum of binary variables,

(iii) restricting the sum of binary variables using auxiliary constraints.

In order to avoid such difficulties, **Chapter 3** describes the application of some numerical methods, namely interpolating polynomial methods for multiple choice parameters. Interpolating polynomials are formulated for all the multi-choice parameters. The multi-choice parameters are replaced by corresponding interpolating polynomials with integer restrictions to formulate a mixed integer NLP problem, which can be solved using nonlinear programming techniques. Formulation as well as solution procedure for solving multi-choice multi-objective transportation problem are also explained with a suitable numerical example.

Chapter 4 implements the transformation techniques to probabilistic programming problem. It presents a probabilistic linear programming problem, where the right hand side parameters are multi-choice type and rest of the parameters are normally distributed independent random variables with known mean and variance in the probabilistic constraints. The model is transformed to an equivalent deterministic mixed integer type NLP by applying MCP methods. The resulting model is then solved by standard nonlinear programming techniques. The chapter also considers a multi-objective probabilistic programming problem involving multi-choice parameters in some probabilistic constraints, which are to be satisfied within certain probability limits. Chance constrained programming technique is used to convert the probabilistic programming model to an equivalent deterministic multi-choice and multi-objective MP model. MCP methodology has been used to deal with multi-choice parameters of the model. A goal programming formulation has been suggested in order to solve multi-objective mixed integer NLP model. Numerical examples are provided to illustrate the solution procedures for both the problems.

Chapter 5 is devoted for evaluation of a fuzzy multi-choice linear programming problem, where some of the parameters and decision variables are trapezoidal type fuzzy numbers. In order to defuzzify a general fuzzy number the concept of nearest trapezoidal fuzzy number is introduced. By assuming all the decision variables as trapezoidal fuzzy number, the objective function and the left hand side of the constraints are approximated to their nearest trapezoidal fuzzy number. Interpolating polynomials are formulated for all the multi-choice type parameters. Multi-choice type parameters are replaced by polynomials with integer restrictions. Then an equivalent multi-objective mixed integer NLP has been established. First and second objective functions represent left and right modal values of the trapezoidal type fuzzy number, where as the third and fourth objective functions represent the left and right spreads of the trapezoidal type fuzzy number, respectively. By applying lexicographic method optimal solution is obtained. In addition, an illustrative example is presented to demonstrate the solution procedure.

Chapter 6 develops a multi-choice multi-objective linear programming model for solving an integrated production planning problem for a steel plant. Integrated production planning aims to integrate the planning sub-functions into a single planning operations. The sub-functions are formulated considering the capacity of different plant units, cost of raw materials from various territories, demands of customers in different geographical locations, time constraints for production of products, production costs, and production rate at different stages of production process. Departure costs are also included during formulation of MP model. Some parameters are decided from a set of possible choices, therefore such parameters are considered as multi-choice type to formulate a multi-choice and multi-objective linear programming model. Multi-choice and multi-objective linear programming models cannot be solved directly. Therefore an equivalent multi-objective MP model has been formulated in order to find the optimal solution of such a complex optimization problem. Computations of the MP model has been performed with the real production data to find the efficiency of the methodology.

Finally, the concluding remarks on the work carried out in Chapter 2 to 6 are presented in Chapter 7. Further scope of future research on the topic is also presented.