Abstract

The present thesis deals with some problems and their solutions associated with meromorphic univalent functions with nonzero pole. We first consider the class $\mathcal{A}(p)$ which consists of functions f that are holomorphic in the open unit disk $\mathbb{D}$ punctured at a point $p \in (0,1)$ where f has a simple pole and $\Sigma(p) := \{f \in \mathcal{A}(p) : f \text{ is univalent}\}$. Then we prove a sufficient condition for these functions to be univalent in $\mathbb{D}$. By using this condition, we construct the family $\mathcal{U}_p(\lambda)$ of all functions $f \in \mathcal{A}(p)$ such that $|(z/f(z))^2 f'(z) - 1| < \lambda\mu$ where $\mu := ((1-p)/(1+p))^2$ for some $0 < \lambda \leq 1$, $z \in \mathbb{D}$. We present some basic results such as obtaining characterization and finding necessary and sufficient coefficient conditions for functions to be in the class $\mathcal{U}_p(\lambda)$. We also obtain a sharp estimate for the Fekete-Szegö functional defined on the class $\mathcal{U}_p(\lambda)$ along with a subordination result for functions in this family.

Next, we obtain the exact value for $\max\limits_{f\in\Sigma(p)} \Delta(r, z/f)$ where $\Delta(r, z/f)$, $0 < r \leq 1$, denotes the Dirichlet integral of the function z/f. We also determine a sharp estimate for $\Delta(r, z/f)$ whenever f belongs to certain subclasses of $\Sigma(p)$. Furthermore, we obtain sharp estimates for the integral means for these classes of functions.

Afterwards, we improve the sufficient condition for univalence for functions in $\mathcal{A}(p)$ with replacing the number μ (defined in the first paragraph) by 1 and subsequently the class of functions $\mathcal{V}_p(\lambda) =: \{f \in \mathcal{A}(p) : |U_f(z)| < \lambda\}$, $\lambda \in (0,1]$ is being introduced. Also we establish that $\mathcal{U}_p(\lambda) \subsetneq \mathcal{V}_p(\lambda)$ and as a consequence, we obtain that all the results proved for the class $\mathcal{U}_p(\lambda)$ will also be valid for the class $\mathcal{V}_p(\lambda)$. Next, we generalize the Area theorem proved by P.N. Chichra for functions in the class $\Sigma(p)$. We also prove an interesting consequence of this result.

Further, we consider the Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$, $|z| < p$ for $f \in \mathcal{V}_p(\lambda)$. Thereafter we find the exact region of variability of $a_2(f)$ for functions in this class. We also prove sharp upper bounds of $|a_n(f)|$, $n \geq 3$ whenever p lies in some subintervals of $(0,1)$. Also we determine non sharp bounds for $|a_n(f)|$, $n \geq 3$ and for $|a_{n+1}(f) - a_n(f)/p|$, $n \geq 2$.

Lastly, we take a closer look on the Laurent expansion of functions $f \in \mathcal{V}_p(\lambda)$ valid in the annulus $0 < |z-p| < 1-p$ and determine the exact region of variability of the residue of f at $z = p$ and find the sharp bounds of the modulus of some initial Laurent coefficients for certain range of values of p. The growth and distortion results for functions in $\mathcal{V}_p(\lambda)$ are also obtained. Next, we prove that $\mathcal{V}_p(\lambda)$ does not contain the class of concave univalent functions for $\lambda \in (0,1]$ and vice-versa for $\lambda \in ((1-p^2)/(1+p^2), 1]$. Finally, we show that there are some sets of values of p and λ for which

$\widehat{\mathbb{C}} \setminus {k_p}^{\lambda}(\mathbb{D})$ may or may not be a convex set, where $\widehat{\mathbb{C}}$ denote the extended complex plane.

Keywords: Analytic functions, Meromorphic functions, Univalent functions, Starlike functions, Convex functions, Concave functions, Meromorphically starlike functions, Subordination, Taylor coefficients, Laurent coefficients, Coefficient bounds, Dirichlet integral, Integral mean, Growth theorem, Distortion theorem.